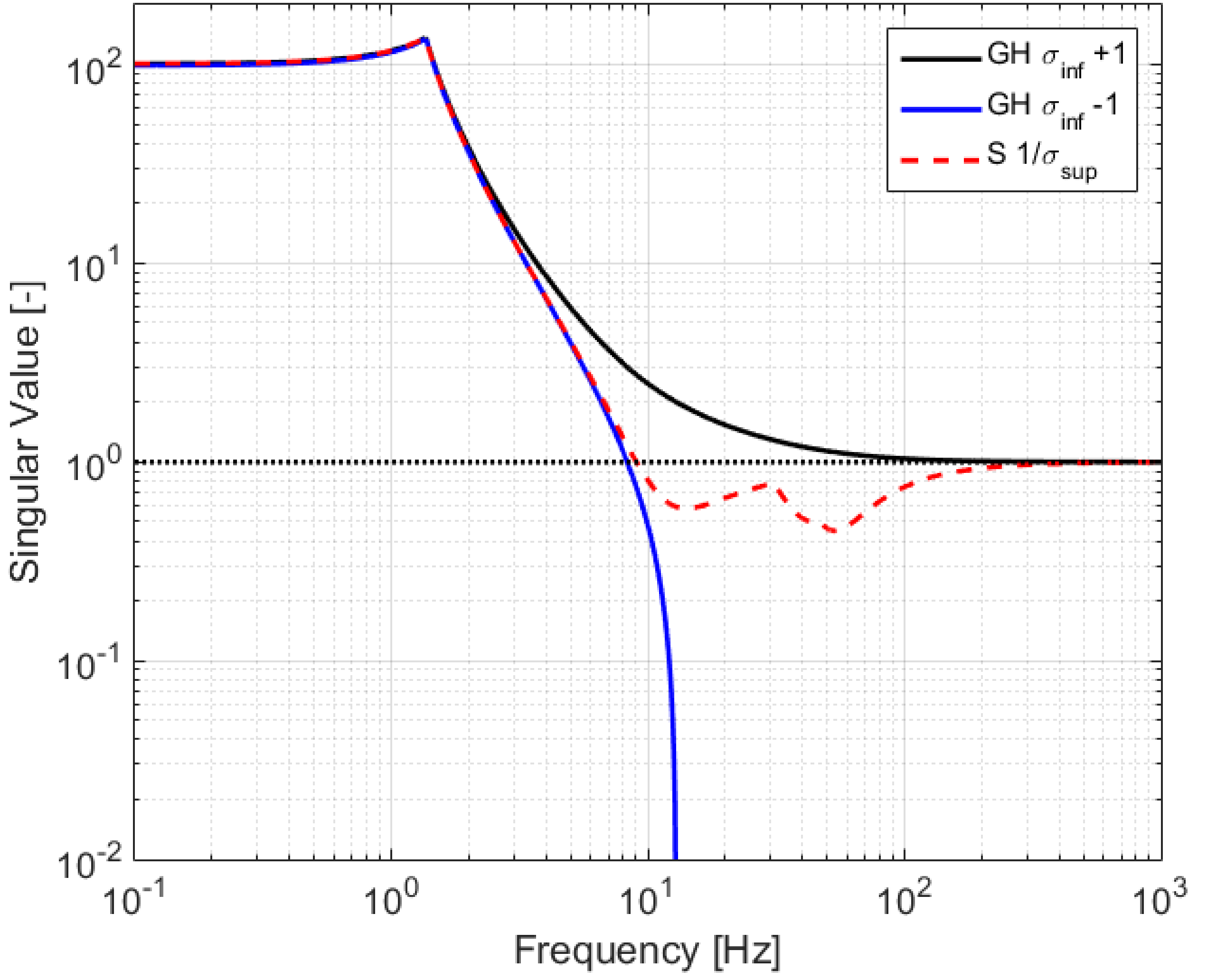
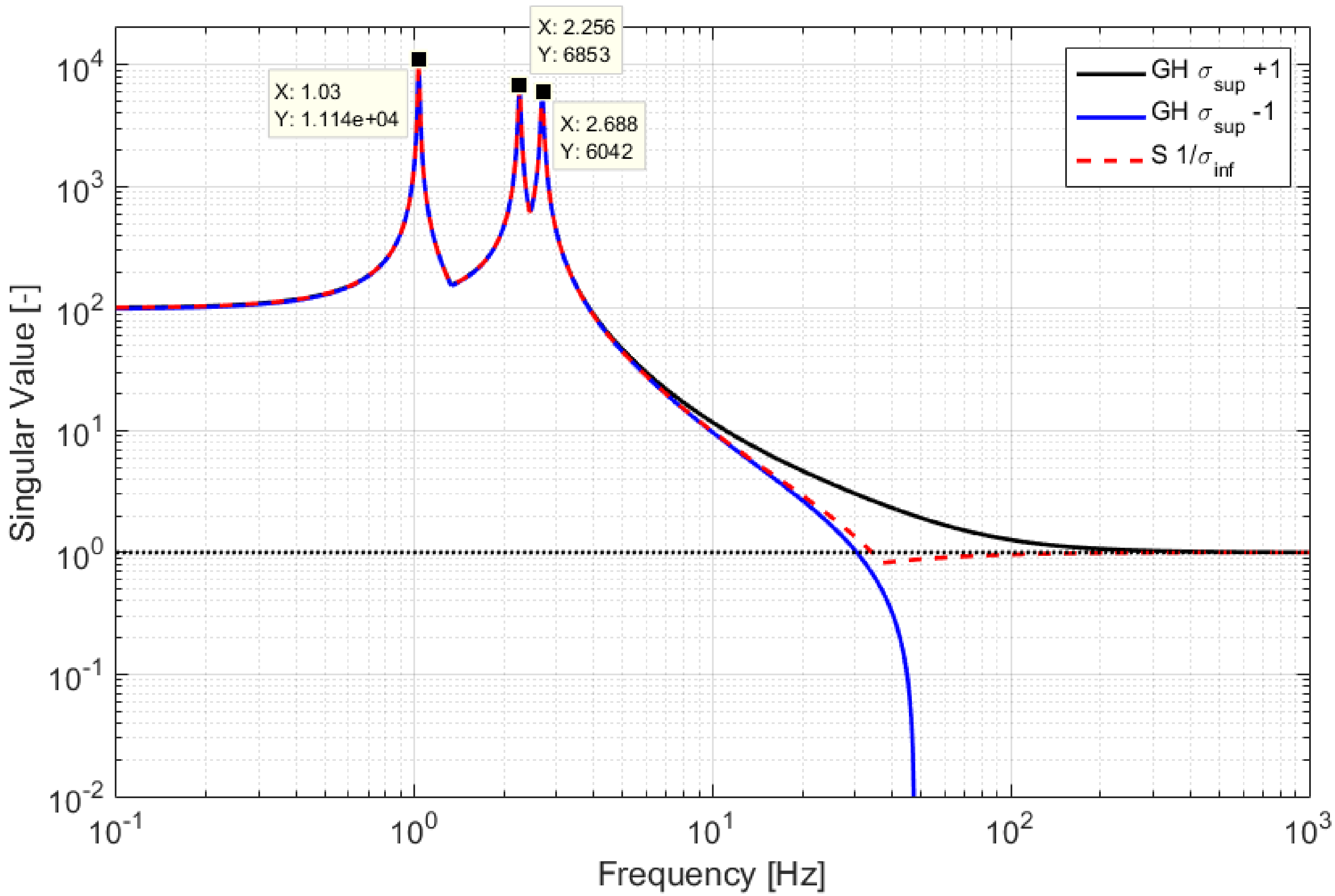
Performance indicator

$$\underline{\sigma}(GH) - 1 \leq \frac{1}{\overline{\sigma}(S)} \leq \underline{\sigma}(GH) + 1$$



Performance indicator

Can we then also say that $\overline{\sigma}(GH) - 1 \leq \frac{1}{\underline{\sigma}(S)} \leq \overline{\sigma}(GH) + 1$?



Performance indicator

Can we then also say that $\overline{\sigma}(GH) - 1 \leq \frac{1}{\underline{\sigma}(S)} \leq \overline{\sigma}(GH) + 1$? → YES

Chapter 29

Some Properties of Singular Values

We list here some important properties of singular values. We leave the proofs to the reader. Some of the properties require that the matrix be square and nonsingular.

1. $\sigma_{max}(M) = \max_{\|x\|_2=1} \|Mx\|_2 = \|M\|_2 = \frac{1}{\sigma_{min}(M^{-1})}$.
2. $\sigma_{min}(M) = \min_{\|x\|_2=1} \|Mx\|_2 = \frac{1}{\|M^{-1}\|_2} = \frac{1}{\sigma_{max}(M^{-1})}$.
3. $\sigma_i(M) - 1 \leq \sigma_i(I + M) \leq \sigma_i(M) + 1, \quad i = 1, \cdots, k$.
4. $\sigma_i(\alpha M) = |\alpha| \sigma_i(M)$ for all $\alpha \in C, i = 1, \cdots, k$.
5. $\sigma_{max}(M_1 + M_2) \leq \sigma_{max}(M_1) + \sigma_{max}(M_2)$.
6. $\sigma_{max}(M_1 M_2) \leq \sigma_{max}(M_1) \cdot \sigma_{max}(M_2)$.

Levine, W. S. (1996). The control handbook. *CRC press.*

1st property

$$\frac{1}{\overline{\sigma}(S)} = \frac{1}{\overline{\sigma}((I + GH)^{-1})} = \underline{\sigma}(I + GH)$$

$$\rightarrow \underline{\sigma}(GH) - 1 \leq \underline{\sigma}(I + GH) \leq \underline{\sigma}(GH) + 1$$

$$\underline{\sigma}(GH) - 1 \leq \frac{1}{\overline{\sigma}(S)} \leq \underline{\sigma}(GH) + 1$$

2nd property

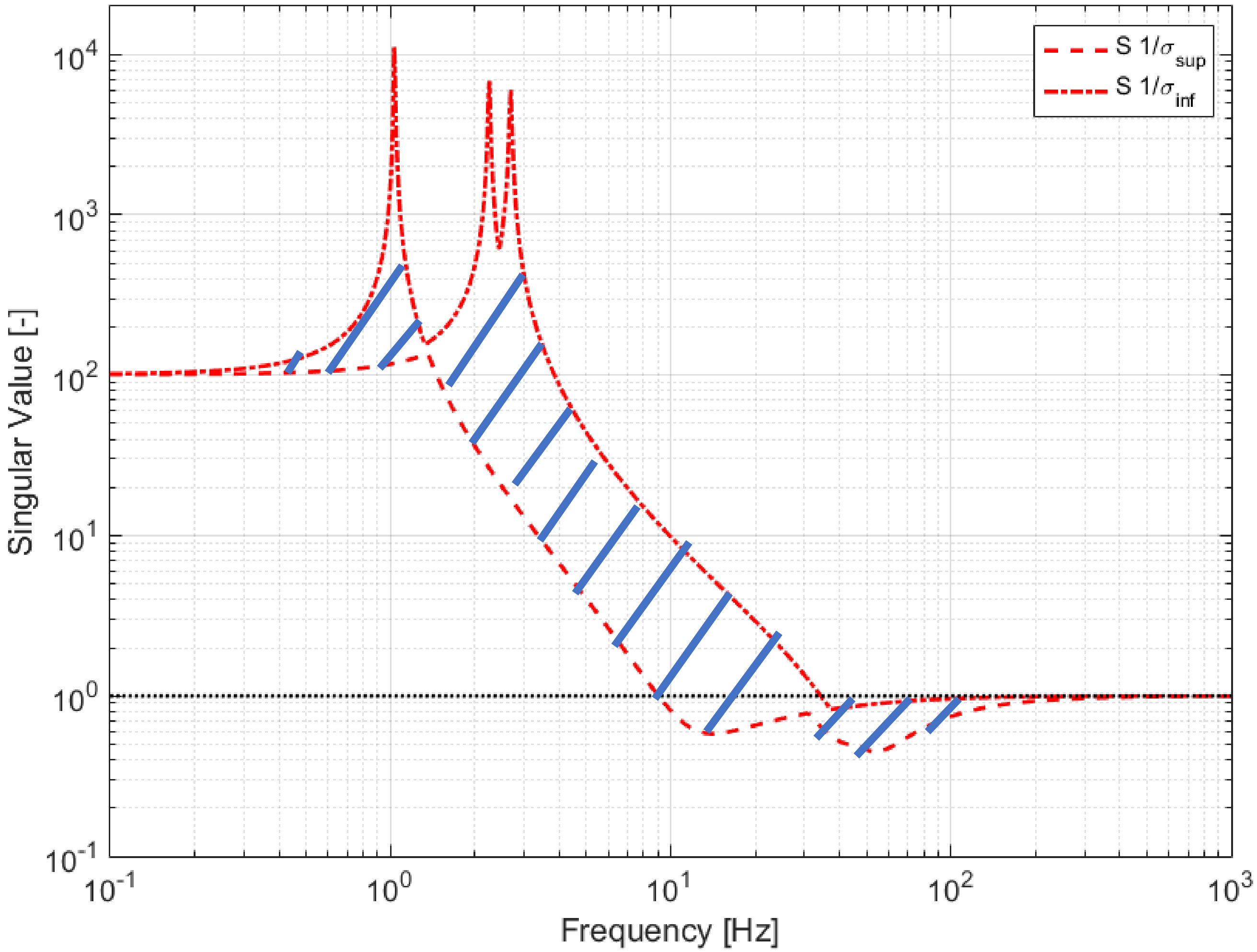
$$\frac{1}{\underline{\sigma}(S)} = \frac{1}{\underline{\sigma}((I + GH)^{-1})} = \overline{\sigma}(I + GH)$$

$$\rightarrow \overline{\sigma}(GH) - 1 \leq \overline{\sigma}(I + GH) \leq \overline{\sigma}(GH) + 1$$

$$\overline{\sigma}(GH) - 1 \leq \frac{1}{\underline{\sigma}(S)} \leq \overline{\sigma}(GH) + 1$$

Performance indicator

As for SISO systems we define the bandwidth as the frequency up to which feedback is effective. For MIMO systems the bandwidth will depend on directions, and we have a *bandwidth region* between a lower frequency where the maximum singular value, $\bar{\sigma}(S)$, reaches 0.7 (the low-gain or worst-case direction), and a higher frequency where the minimum singular value, $\underline{\sigma}(S)$, reaches 0.7 (the high-gain or best direction). If we want to associate a



It seems like here we show the maximum and minimum performance and to achieve these performance, we have to make a change of coordinates to be in the direction of these singular values.

Meaning of the zeros in a MIMO system

Transmission zeros → explain my understanding when time

24.4.1 Definition of MIMO Transmission Zeros

To define the multi-input, multioutput (MIMO) transmission zeros, we will first assume that we have a system with the same number of inputs and outputs. This is referred to as a square system. We will later extend the definition to nonsquare systems. For square systems, we can represent the system in the time domain as

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (24.36)$$

$$y(t) = Cx(t) + Du(t) \quad (24.37)$$

where $x(t) \in \mathcal{R}^n$, $u(t) \in \mathcal{R}^m$, and $y(t) \in \mathcal{R}^m$. We can also write the transfer function matrix as

$$G(s) = C(sI - A)^{-1}B + D \quad (24.38)$$

where $G(s) \in \mathcal{C}^{m \times m}$. Given this system, we have the following definition:

DEFINITION 24.3 The plant has a zero at the (complex) value z_k if vectors $\xi_k \in \mathcal{C}^n$ and $u_k \in \mathcal{C}^m$ exist which are not both zero, so that the solution to the equations

$$\dot{x}(t) = Ax(t) + Bu_k e^{z_k t}, \quad x(0) = \xi_k \quad (24.39)$$

$$y(t) = Cx(t) + Du(t) \quad (24.40)$$

has the property that

$$y(t) \equiv 0 \quad \forall t > 0 \quad (24.41)$$

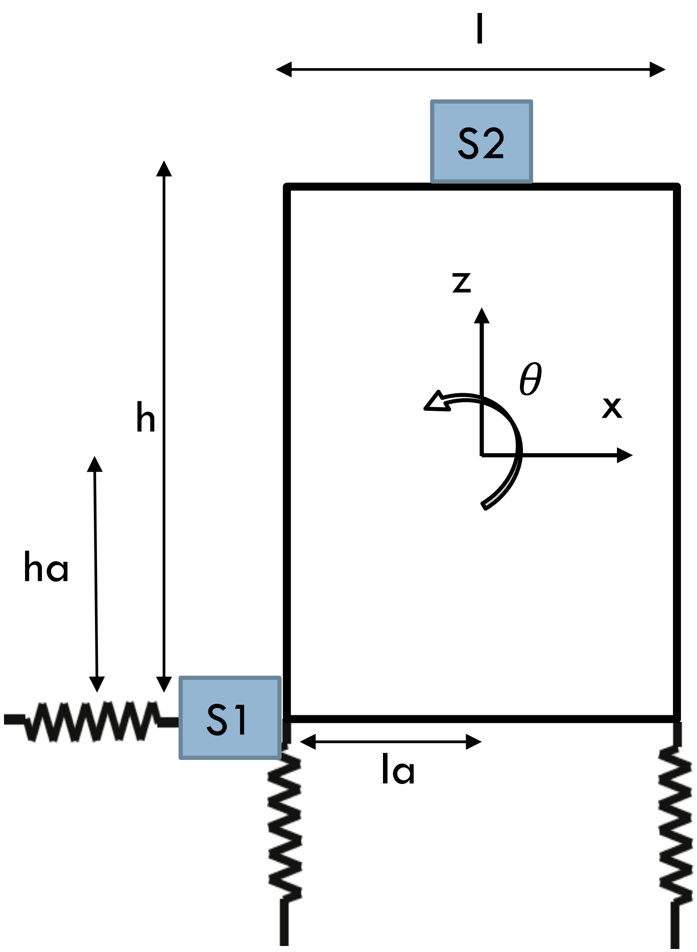
This property of transmission zeros is sometimes called transmission blocking. When zeros repeat, this definition still holds but a more complicated transmission blocking property also holds [2].

As an example, let us show that this definition is consistent with the standard definition of zeros for single-input, single-output systems.

at the same frequency as the pole at $s = -3$. It is also important to realize that, although one of the SISO transfer functions has a zero at $s = -2$, this is not a transmission zero of the MIMO system.

Levine, W. S. (1996). The control handbook. *CRC press,* chap. 24
Provides also information about pole-zero cancellation

3 dof model with 3 inputs and 4 outputs



Study SVD decomposition

Compare isolation performance using SVD and with a decentralized control

SVD decomposition for a lxm matrix

Consider a fixed frequency ω where $G(j\omega)$ is a constant $l \times m$ complex matrix, and denote $G(j\omega)$ by G for simplicity. Any matrix G may be decomposed into its singular value decomposition, and we write

$$G = U\Sigma V^H \tag{3.31}$$

where

Σ is an $l \times m$ matrix with $k = \min\{l, m\}$ non-negative singular values, σ_i, arranged in descending order along its main diagonal; the other entries are zero. The singular values are the square roots of the eigenvalues of $G^H G$, where G^H is the complex conjugate transpose of G.

$$\sigma_i(G) = \sqrt{\lambda_i(G^H G)} \tag{3.32}$$

U is an $l \times l$ unitary matrix of output singular vectors, u_i,

V is an $m \times m$ unitary matrix of input singular vectors, v_i,

SVD decomposition for a lxm matrix

Maximum and minimum singular value. As already stated, it can be shown that the largest gain for *any* input direction is equal to the maximum singular value

$$\max_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \frac{\|Gv_1\|_2}{\|v_1\|_2} = \sigma_1(G) \triangleq \bar{\sigma}(G) \tag{3.38}$$

and that the smallest gain for any input direction is equal to minimum singular value

$$\min_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \frac{\|Gv_k\|_2}{\|v_k\|_2} = \sigma_k(G) \triangleq \underline{\sigma}(G) \tag{3.39}$$

where $k = \min\{l, m\}$. Thus, for any vector d we have that

$$\underline{\sigma}(G) \leq \frac{\|Gd\|_2}{\|d\|_2} \leq \bar{\sigma}(G) \tag{3.40}$$

Define $u_1 = \bar{u}, v_1 = \bar{v}, u_k = \underline{u}$ and $v_k = \underline{v}$. Then it follows that

$$G\bar{v} = \bar{\sigma}\bar{u}, \qquad G\underline{v} = \underline{\sigma}\ \underline{u} \tag{3.41}$$

The vector $\bar{v}$ corresponds to the input direction with largest amplification, and $\bar{u}$ is the corresponding output direction in which the inputs are most effective. The direction involving $\bar{v}$ and $\bar{u}$ is sometimes referred to as the "strongest", "high-gain" or "most important" direction. The next most important direction is associated with v_2 and u_2, and so on (see Appendix A.3.6) until the "least important", "weak" or "low-gain" direction which is associated with $\underline{v}$ and $\underline{u}$.

SVD decomposition for a lxm matrix

Nonsquare plants

The SVD is also useful for nonsquare plants. For example, consider a plant with 2 inputs and 3 outputs. In this case the third output singular vector, u_3, tells us in which output direction the plant cannot be controlled. Similarly, for a plant with more inputs than outputs, the additional input singular vectors tell us in which directions the input will have no effect.

Here, we have 4 outputs and 3 inputs but we are in a 3 dof system

→ the system is fully controllable and fully observable

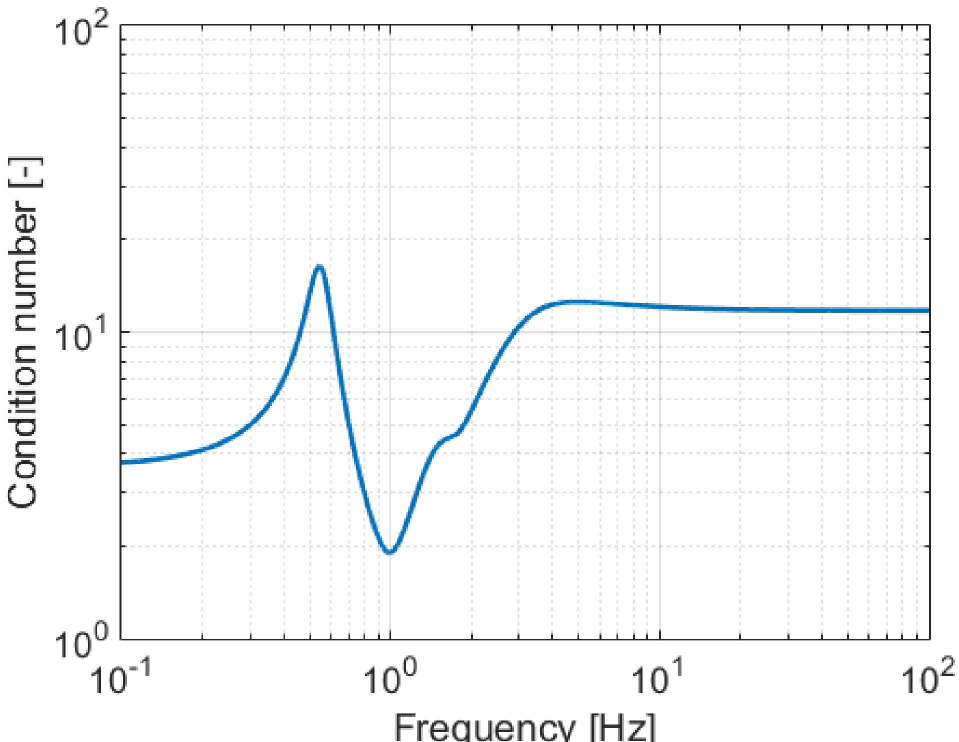
We are going to decompose the system at the crosseover frequency (around 10 Hz)

Performance indicator 1: Condition number

We define the condition number of a matrix as the ratio between the maximum and minimum singular values

$$\gamma(G) = \bar{\sigma}(G)/\underline{\sigma}(G) \tag{3.63}$$

A matrix with a large condition number is said to be *ill-conditioned*. For a nonsingular (square) matrix $\underline{\sigma}(G) = 1/\bar{\sigma}(G^{-1})$, so $\gamma(G) = \bar{\sigma}(G)\bar{\sigma}(G^{-1})$. It then follows from (A.117) that the condition number is large if both G and G^{-1} have large elements.



What does « large » condition number mean?

Ex. 3.5: $\gamma(G) > 100$ and they say that this problem is ill- conditioned.

Performance indicator 1: Condition number

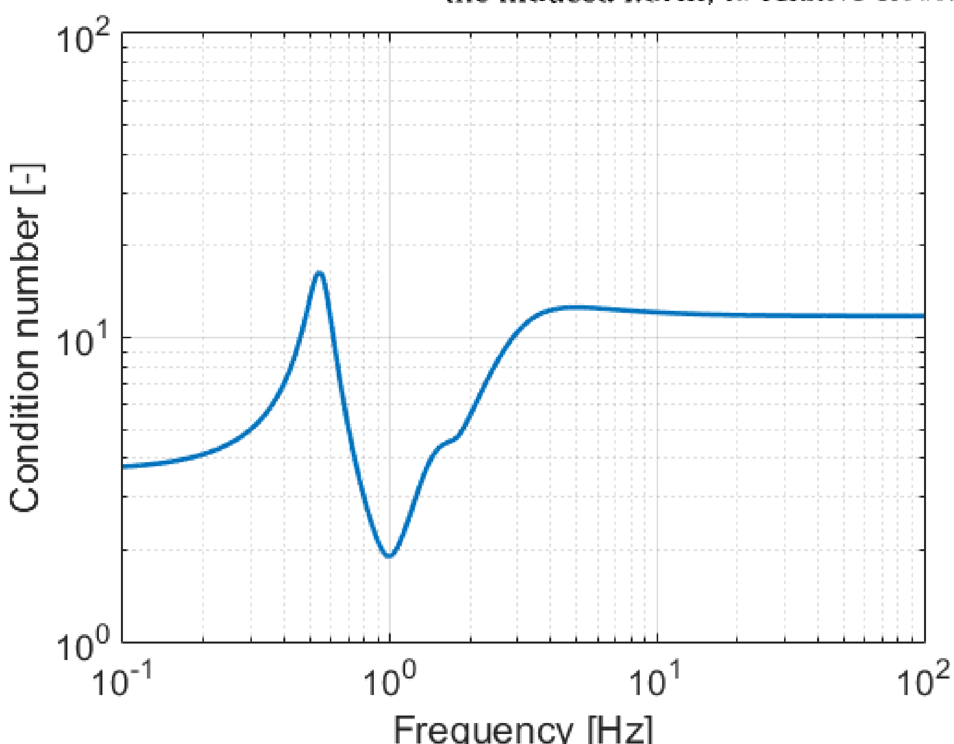
What does « large » condition number mean?

where $\kappa(\mathbf{A})$ denotes the *condition number* of $\mathbf{A}$, defined as

$$\kappa(\mathbf{A}) = \|\mathbf{A}\|\|\mathbf{A}^{-1}\| \qquad (3.60)$$

Since $\kappa(\mathbf{A}) = \|\mathbf{A}\|\|\mathbf{A}^{-1}\| \geq \|\mathbf{A}\mathbf{A}^{-1}\| = 1$, when $\kappa(\mathbf{A}) \approx 1$, the matrix $\mathbf{A}$ is well-conditioned, but when $\kappa(\mathbf{A}) \gg 1$, the matrix $\mathbf{A}$ is ill-conditioned. The condition number of $\mathbf{A}$ also serves as the multiplier scaling relative errors in $\mathbf{A}$, measured by the induced norm, to relative errors in $\mathbf{x}$ [4].

Levine, W. S. (1996). The control handbook. *CRC press,* chap. 3



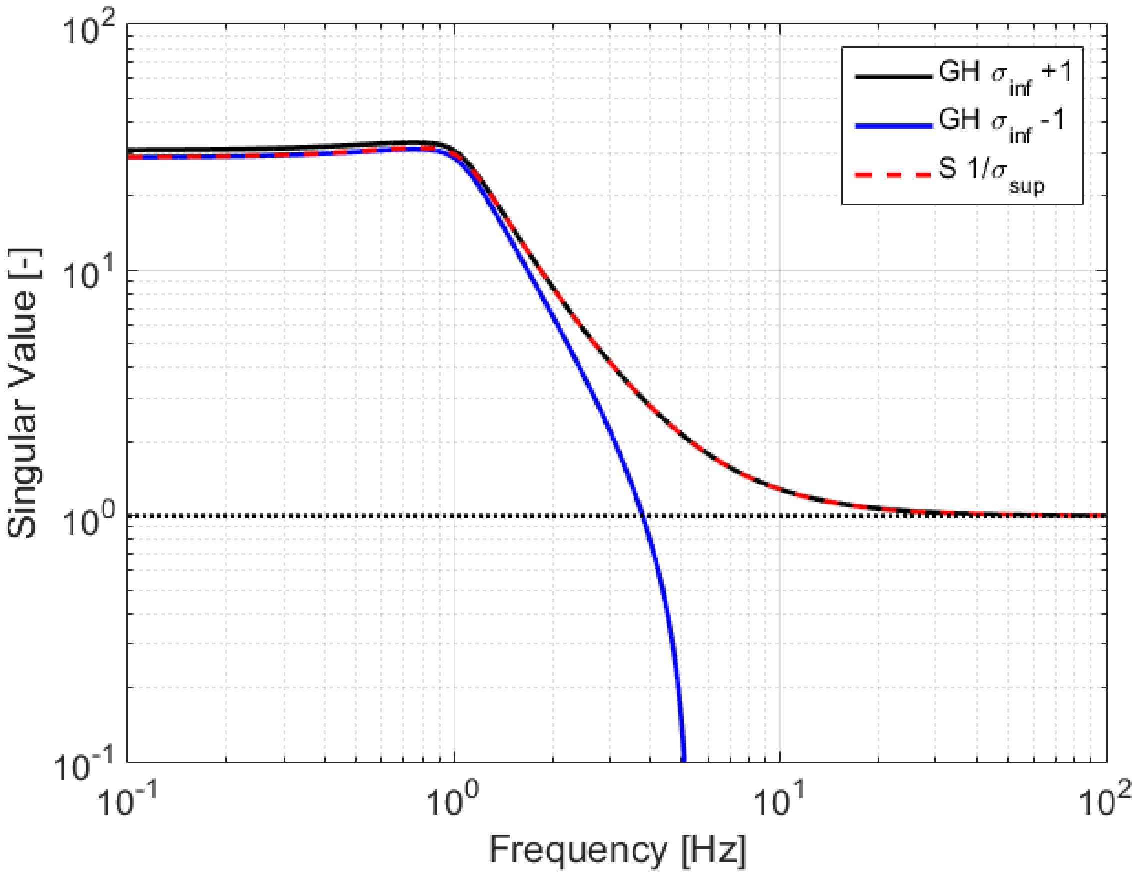
In the field of numerical analysis, the condition number of a function with respect to an argument measures how much the output value of the function can change for a small change in the input argument. This is used to measure how sensitive a function is to changes or errors in the input, and how much error in the output results from an error in the input.

A problem with a low condition number is said to be well-conditioned, while a problem with a high condition number is said to be ill-conditioned.
From wikipedia

Performance indicator 2:
singular values of the open loop

Minimum performance

$$\underline{\sigma}(GH) - 1 \leq \frac{1}{\overline{\sigma}(S)} \leq \underline{\sigma}(GH) + 1$$



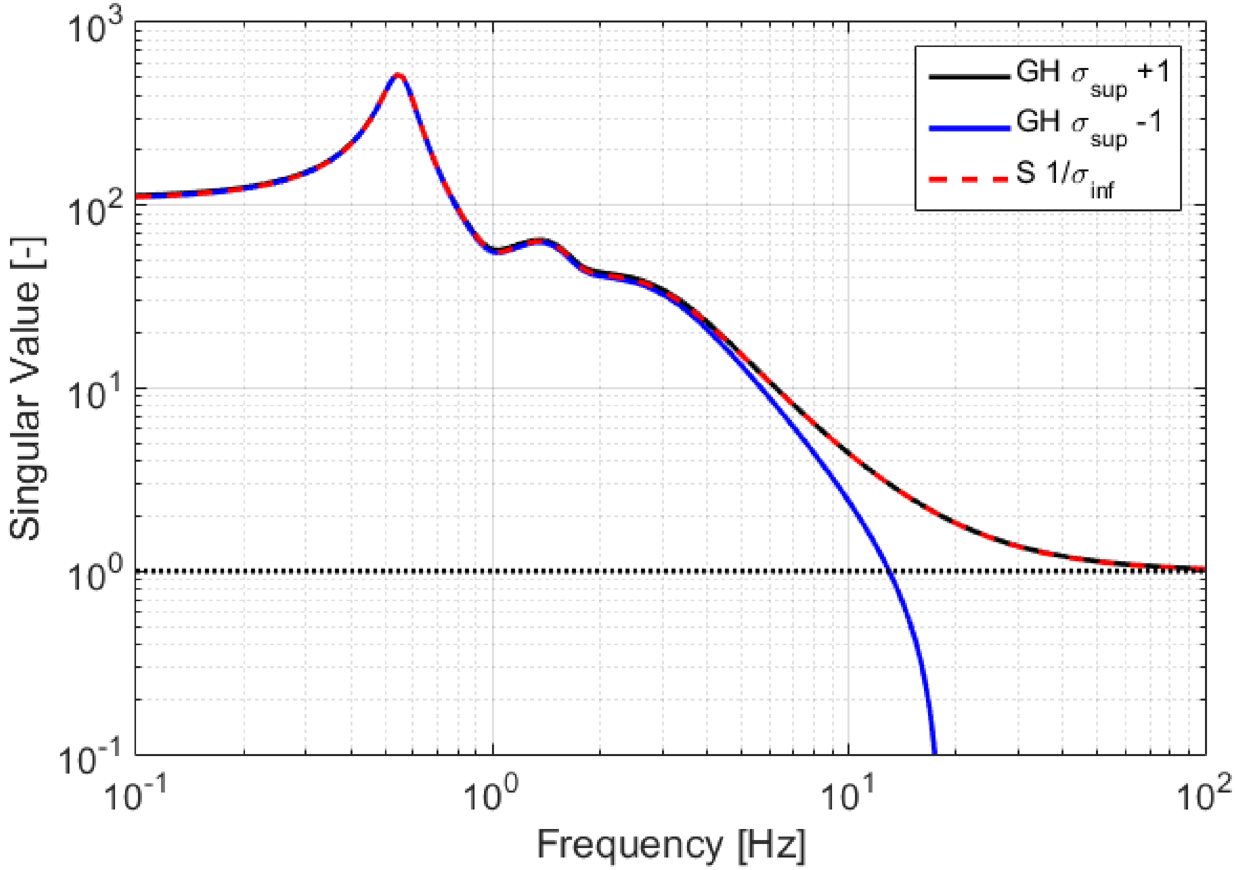
$$H_{svd} = V^{-H}(f_0)HU^{-1}(f_0)$$

Where H is a 3x4 negative identity matrix, i.e. the last column contains only zeros.

Performance indicator 2: singular values of the open loop

Maximum performance

$$\overline{\sigma}(GH) - 1 \leq \frac{1}{\underline{\sigma}(S)} \leq \overline{\sigma}(GH) + 1$$



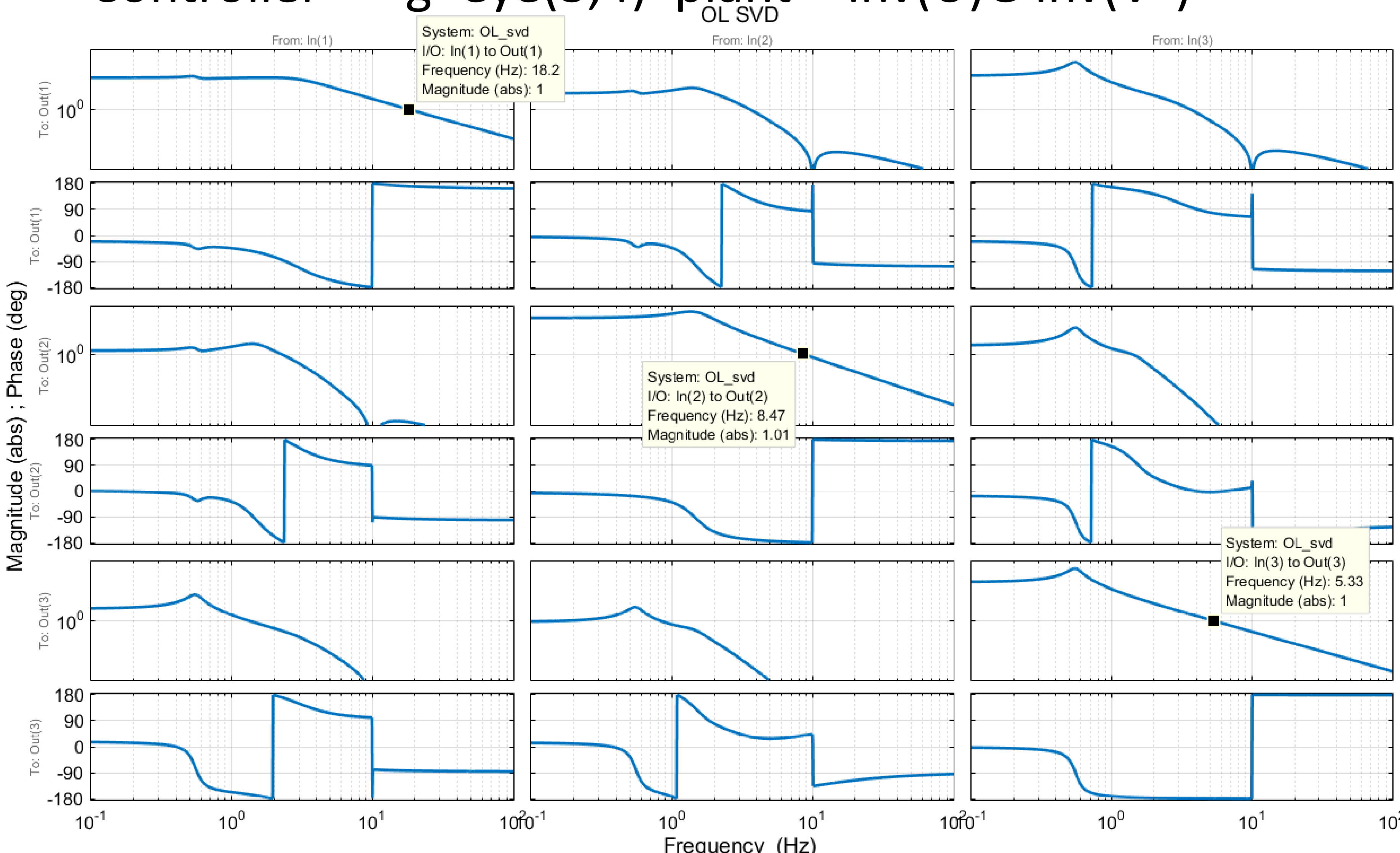
Non-square systems approach

Previous approaches in the literature have solved this problem (non square system) by (1) squaring the system by discarding some inputs or by adding new outputs, or (2) by utilizing some inputs for input/output (I/O) linearization and the remaining inputs for minimizing cost.

Kolavennu, Soumitri. "ROBUST NONLINEAR CONTROL OF NONSQUARE MULTIVARIABLE SYSTEMS." *Dynamics and Control of Process Systems 2001 (DYCOPS-6): A Proceedings Volume from the 6th IFAC Symposium, Jejudo Island, Korea, 4-6 June 2001*. Vol. 1. Pergamon, 2001.

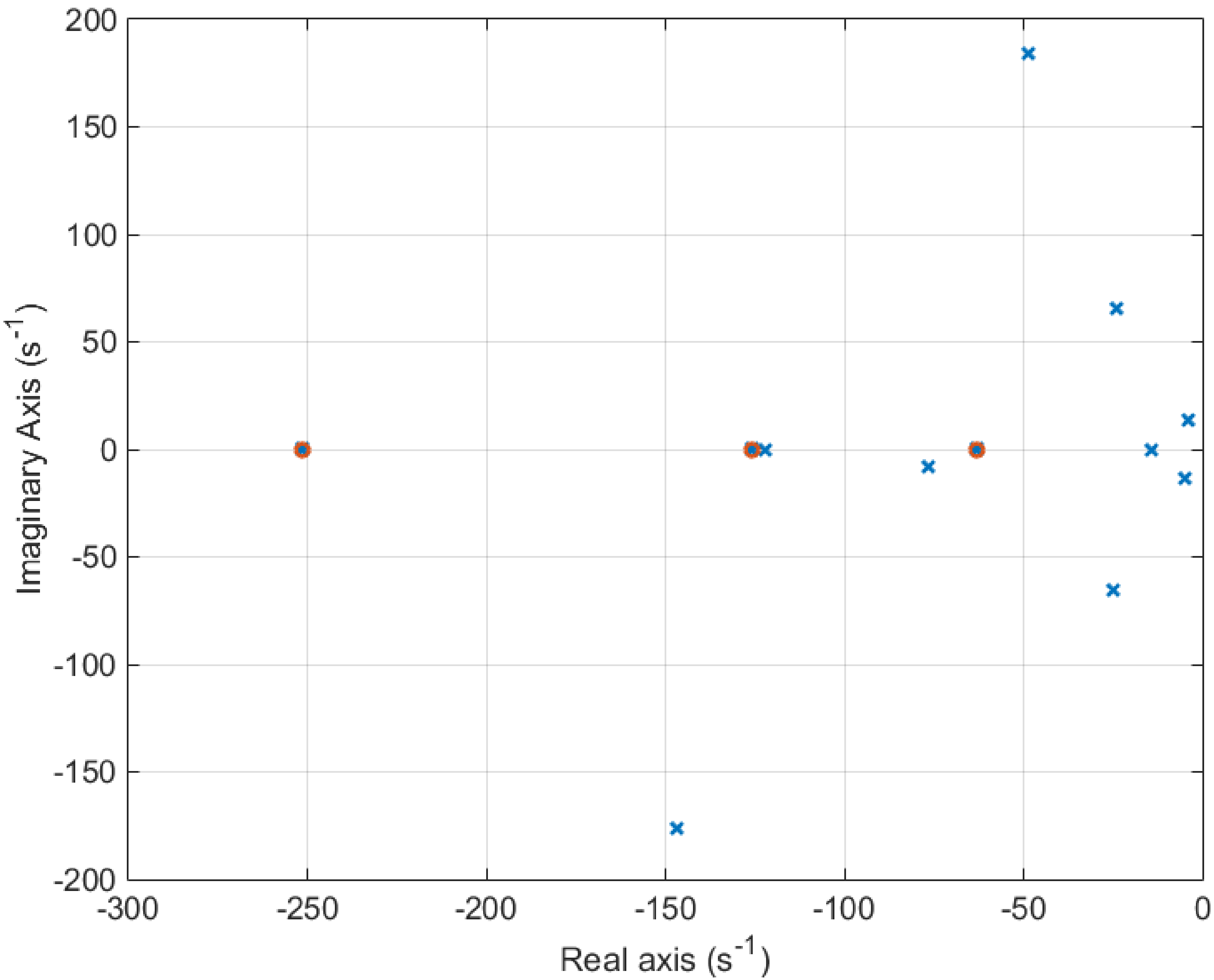
Design of the controller

Controller = - g*eye(3,4) plant = inv(U)G inv(V^H)



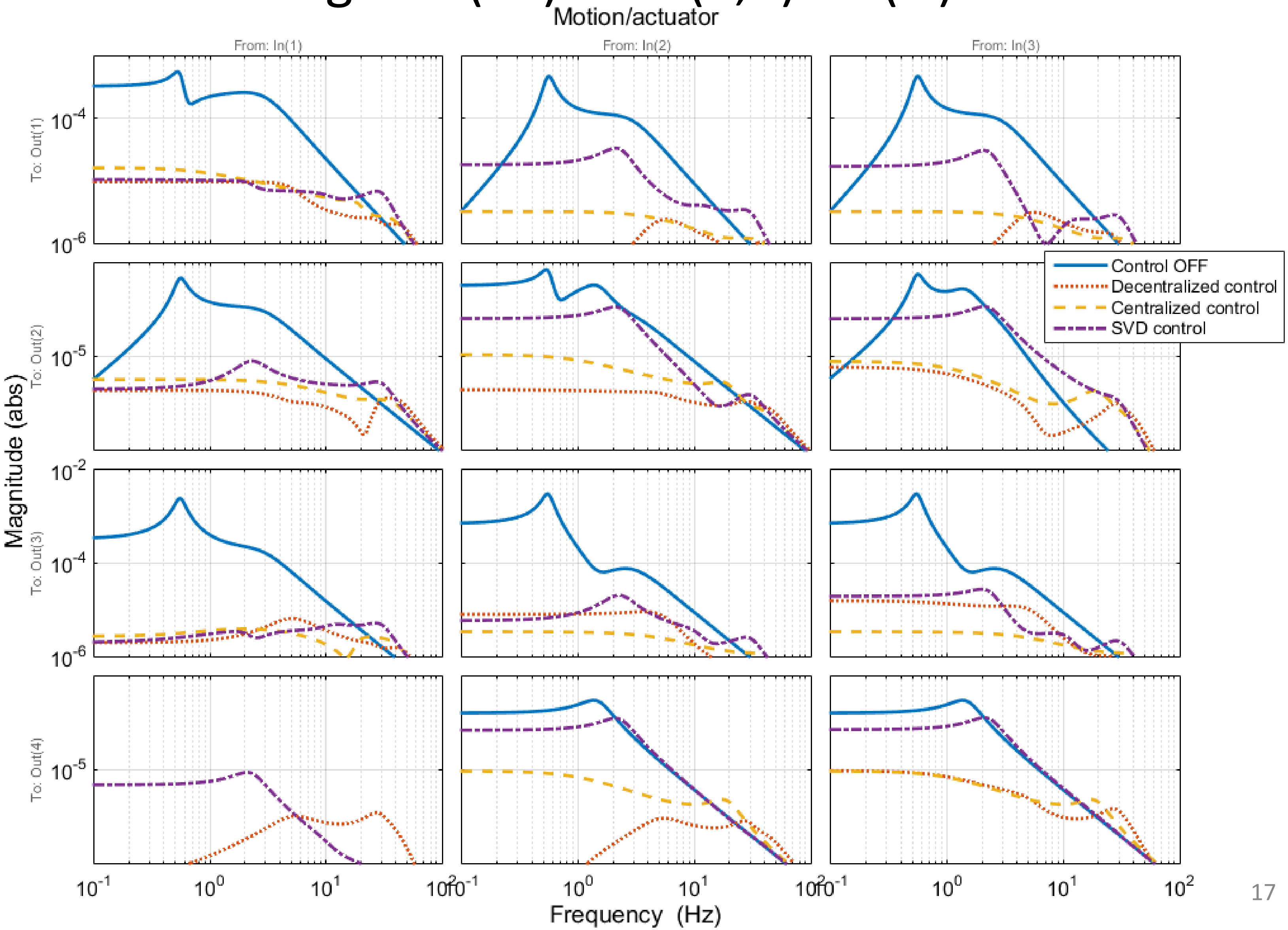
Design of the controller

Controller = - g* inv(V^H) Lead(3,4) inv(U)



Design of the controller

Controller = - g* inv(V^H) Lead(3,4) inv(U)



Design of the controller

The gain is different depending on the singular value direction → we should normalize the three transfer functions by their static gain

1. Compute SVD at the crossover frequency

$$[U, S, V] = svd(system(10\ Hz))$$

2. Compute the static gain

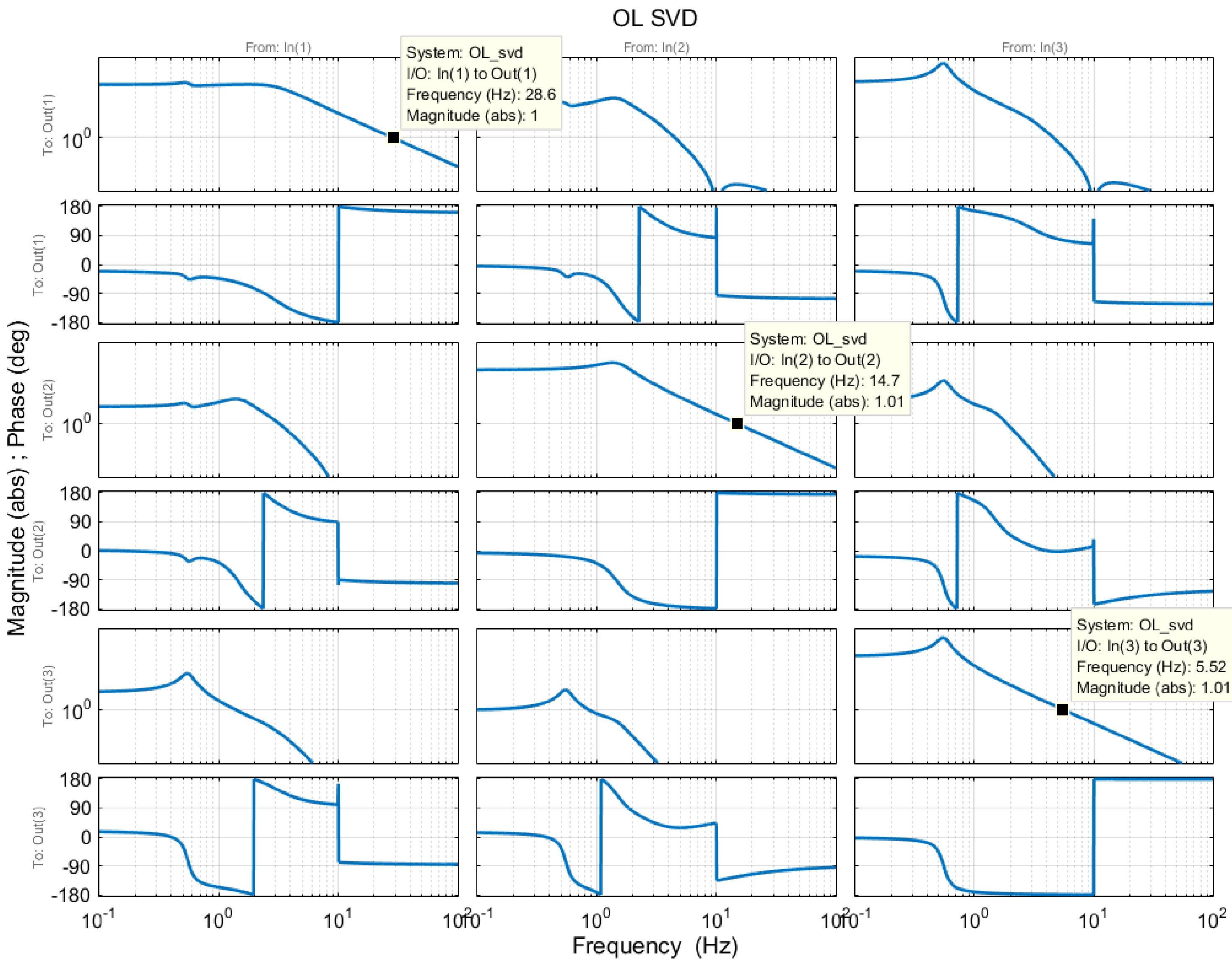
$$c_{norm} = |U^{-1} system(0\ Hz)\ V^{-H}|$$

3. Normalize the controller

$$gainMatrix = \begin{bmatrix} 1/c_{norm}(1,1) & 0 & 0 & 0 \\ 0 & 1/c_{norm}(2,2) & 0 & 0 \\ 0 & 0 & 1/c_{norm}(3,3) & 0 \end{bmatrix}$$

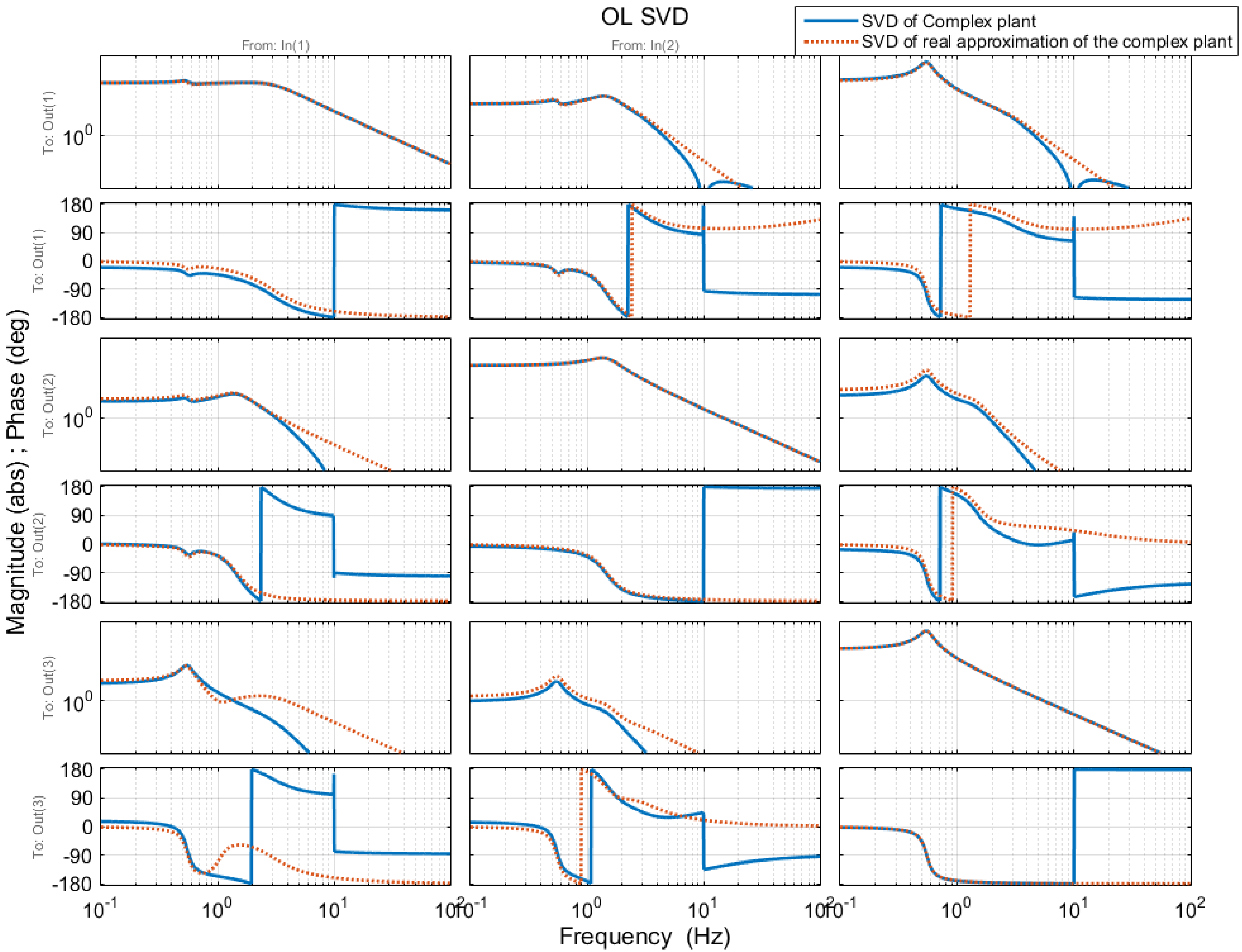
Design of the controller

Controller = - 100*gainMatrix



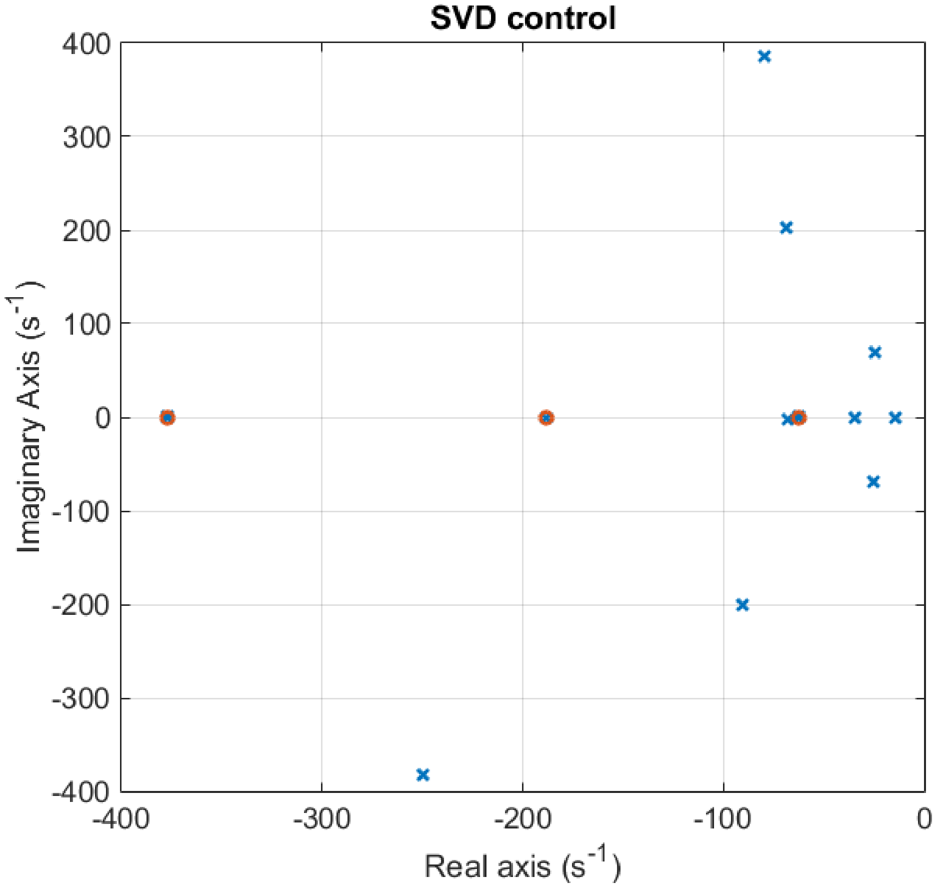
Design of the controller

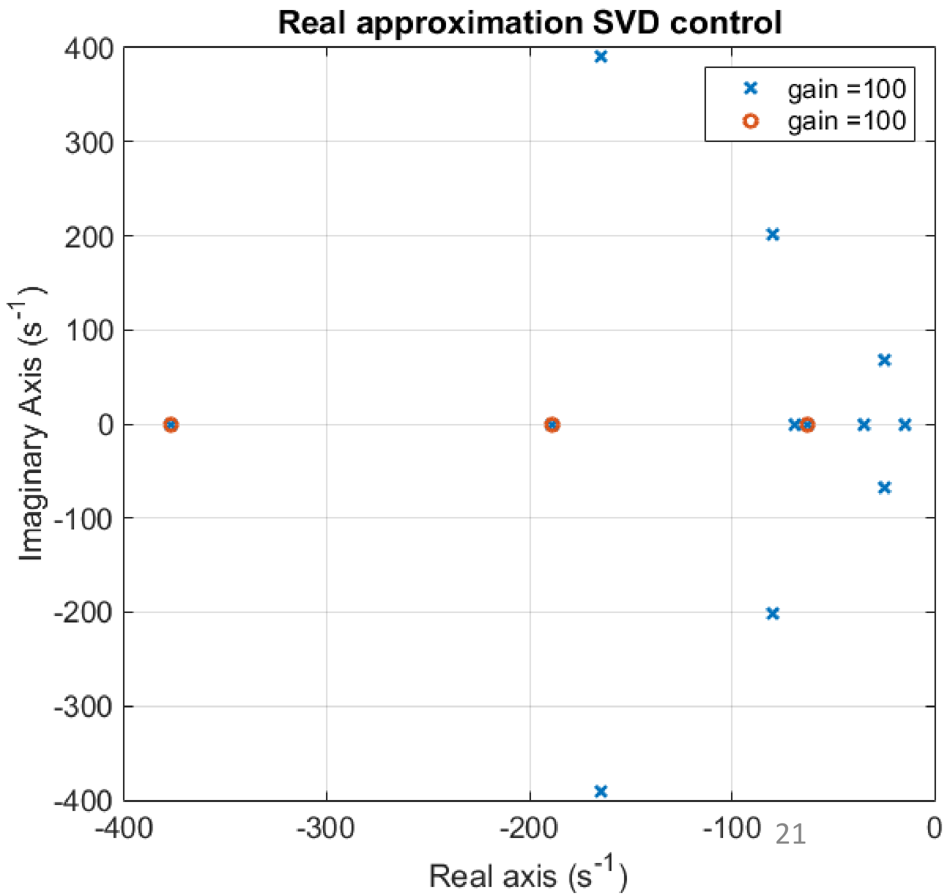
Controller = - 100*gainMatrix



Design of the controller

$$H = -100\begin{bmatrix} Lead1/c_{norm}(1,1) & 0 & 0 & 0 \\ 0 & Lead2/c_{norm}(2,2) & 0 & 0 \\ 0 & 0 & Lead3/c_{norm}(3,3) & 0 \end{bmatrix}$$





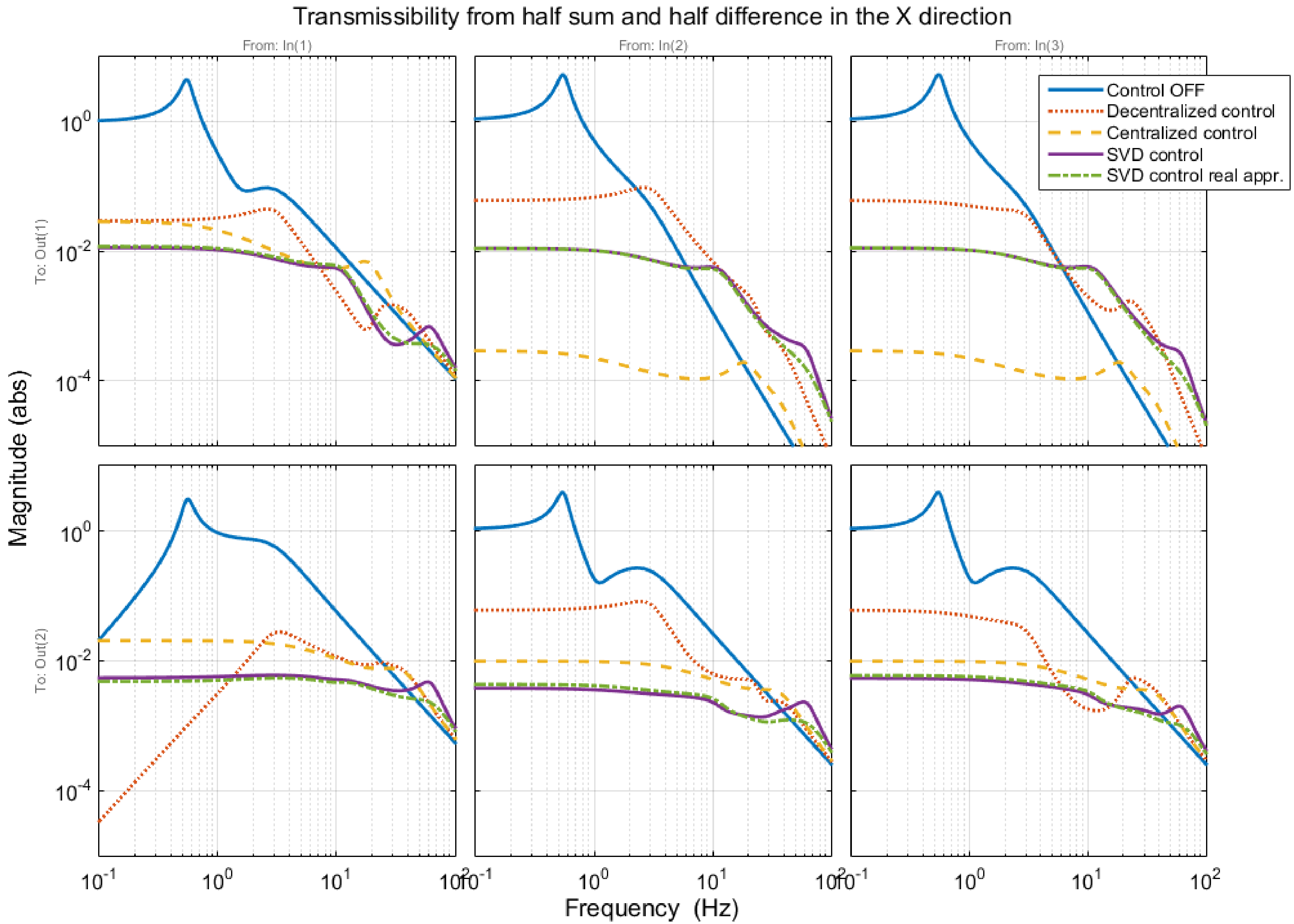
Performance

Half sum

$$\frac{x_1 + x_2}{2}$$

Half difference

$$\frac{x_1 - x_2}{2}$$



Horizontal ground motion

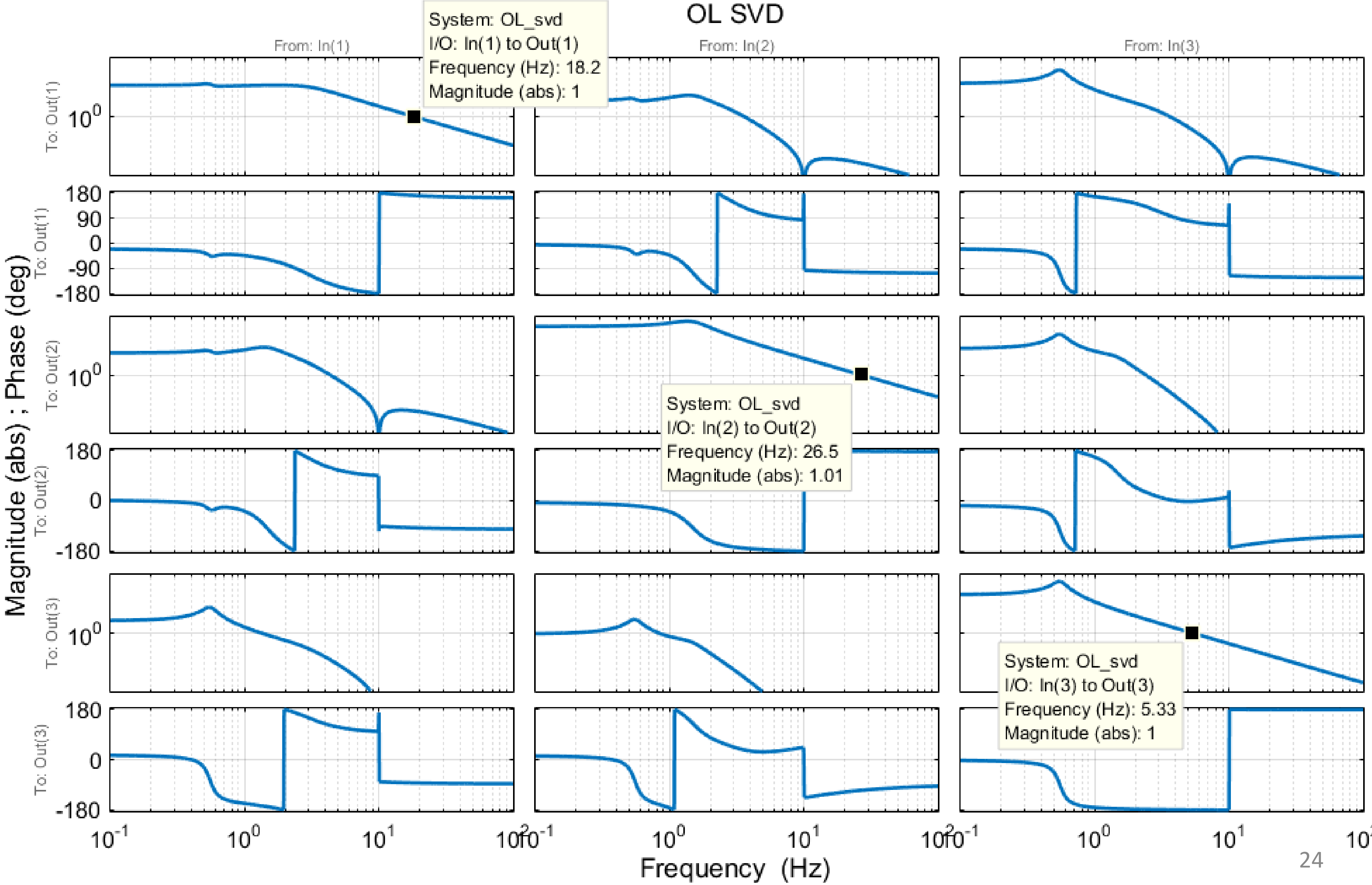
Vertical ground motion (left leg)

Vertical ground motion (right leg)

Old slides

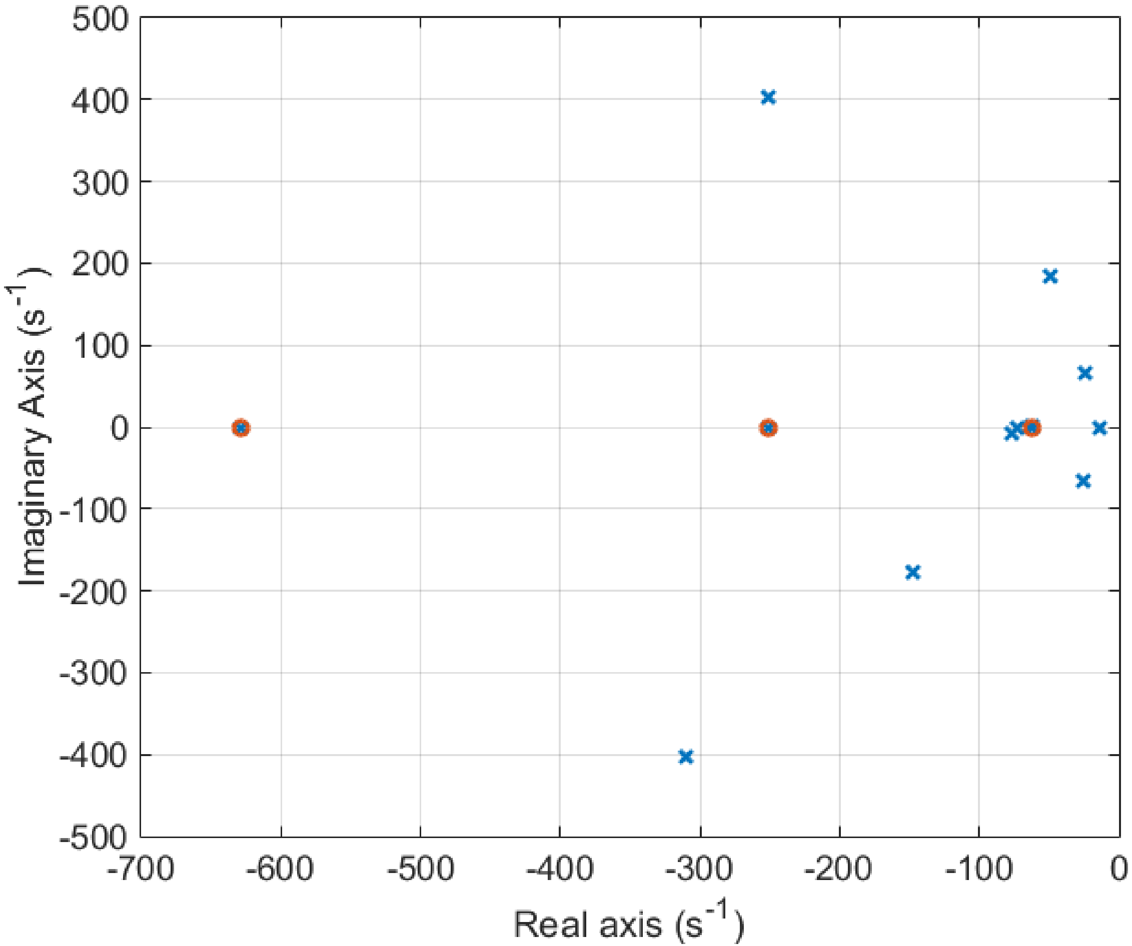
Design of the controller

Controller = - g*[1 0 0 0; 0 10 0 0; 0 0 1 0] plant = inv(U)G inv(V^H)

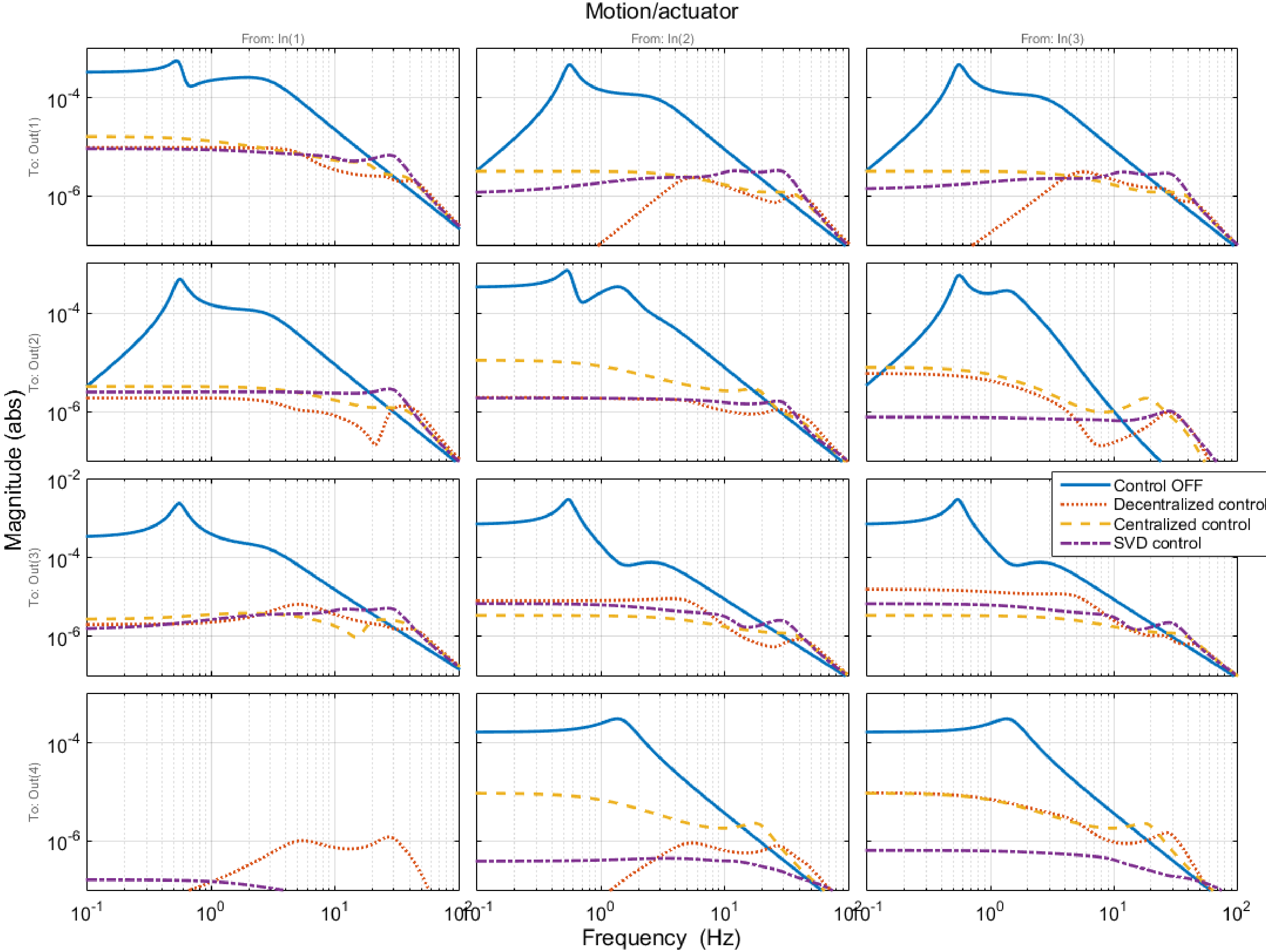


Design of the controller

$$\text{Controller} = -\,g^{*}\,\text{inv}(V^{H}) \begin{bmatrix} Lead1 & 0 & 0 \\ 0 & 10\,Lead2 & 0 \\ 0 & 0 & Lead3 \\ 0 & 0 & 0 \end{bmatrix} \text{inv}(U)$$



Design of the controller



Design of the controller

SVD of the real approximation of the plant at 10 Hz

- The controller is exactly the same as the one designed for the complex plant
- The phase shift of the plant due to the fact that the SVD is done above the resonance frequency is also there → minus sign is needed
- Gershgorin radius: how to apply it in case of non square matrix? → couldn’t find the paper cited by Mohit ☹ + couldn’t find any paper applying this to non-square systems

Design of the controller

SVD of the real approximation of the plant at 10 Hz