Summary of the MIMO meetings

Jennifer Watchi

September 14, 2020

Consider the following MIMO system whose inputs and outputs are in the decentralized frame

$$A = \begin{pmatrix} 0 & I \\ -K/M & -C/M \end{pmatrix},\ B = \begin{pmatrix} 0 \\ B/M \end{pmatrix},\ C = \begin{pmatrix} J & 0 \end{pmatrix},\ D = 0$$

Where J is the Jacobian, B is the transposed of the Jacobian, K is the stiffness matrix, C is the damping matrix, M is the mass matrix and I is an identity matrix.

In the rest of the document, the decentralized system will be defined by G and the controller by H.

This document will explain how to project the decentralized system into centralized, singular values and eigenvalues coordinates and will discuss the pros and cons of these different frames. An example will be used to illustrate the different aspects of MIMO systems discussed during the meetings.

1 Projection of the system in different spaces

1.1 Cartesian decomposition (central coordinates)

Therefore, if you want to project G in the centralized coordinates, you have to apply

$$G_{centralized} = J^{-1}GB^{-1} \tag{1}$$

Note that being in the centralized coordinates does not mean that you will have a decoupled system. In fact, you are expressing the system in the generalized coordinates (x,y,z,θ_x,θ_y,θ_z) whose axis origins correspond to the centre of mass of the system. Consequently, if you have some coupling between two directions, the non-diagonal term corresponding to this coupling will not be null.

If you want to apply a centralized controller to G (not $G_{centralized}$), the controller will be

$$H_{cen} = B^{-1}HJ^{-1} \tag{2}$$

i.e. we project the outputs in the centralized coordinates, we apply the controller and we project the result in the legs coordinates.

1.2 Singular value decomposition (SVD)

“Some advantage of the SVD over the eigenvalue decomposition for analysing gains and directionality of multivariable plants are:

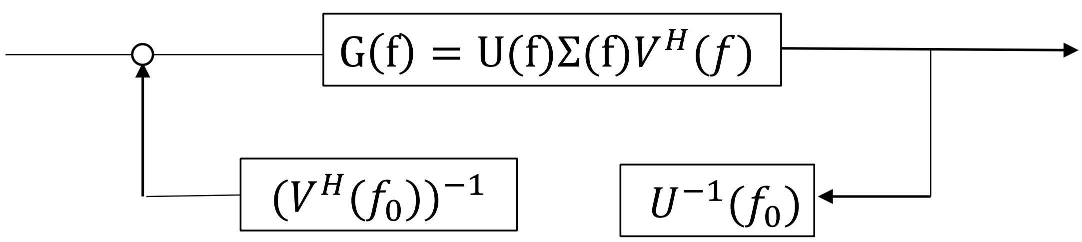
1. The singular values give better information about the gains of the plants
2. The SVD directions obtained from the SVD are orthogonal
3. The SVD also applies directly to non-square plants.”p.77 [1]

A good mathematical definition of the singular values can be found in the appendix A.3 of the same book.

To prove the 1st assumption above, the gain of a MIMO system can be defined as (p.73)

$$
\begin{aligned}
\|G\| &= \sup_{\|d\|\neq 0} \frac{\|Gd\|}{\|d\|} = \sup_{\|d\|\neq 0} \sqrt{\frac{d^H G^H G d}{d^H d}} \\
&= \sup_{\|d\|\neq 0} \sqrt{\frac{d^H (U\Sigma V^H)^H (U\Sigma V^H) d}{d^H d}} \\
&= \sup_{\|d\|\neq 0} \sqrt{\frac{d^H V^H \Sigma^H U^H U \Sigma V^H d}{d^H d}} \\
&= \sup_{\|d\|\neq 0} \sqrt{\frac{d^H V^H \|\Sigma\|^2 V^H d}{d^H d}} \\
&= \sup_{\|d\|\neq 0} \sqrt{\frac{\|\Sigma\|^2 d^H V^H V^H d}{d^H d}} \\
&= \|\Sigma\| \sup_{\|d\|\neq 0} \sqrt{\frac{d^H d}{d^H d}} = \|\Sigma\|
\end{aligned}
$$

Therefore, the gain of a MIMO system corresponds to the singular values and consequently, singular values provide better information about the gain of the plant as stated in the book.



The SVD can be made in different ways (p.92):

1. Dynamic decoupling

The decoupling is done for each frequency.

2 Steady-state decoupling

The decoupling is calculated a 0 Hz and the resulting U and V matrix are used to project G in the directions of the singular values at 0 Hz.

3 Approximate decoupling at frequency f_0

We chose a frequency at which we evaluate the U and V matrix and we use it to project G in the direction of the singular values obtained.

$$G(f_0) = U(f_0)\Sigma(f_0)V^H(f_0)$$
$$\Sigma(f_0) = U^{-1}G(f_0)V^{-H}(f_0)$$

The decoupled system is thus

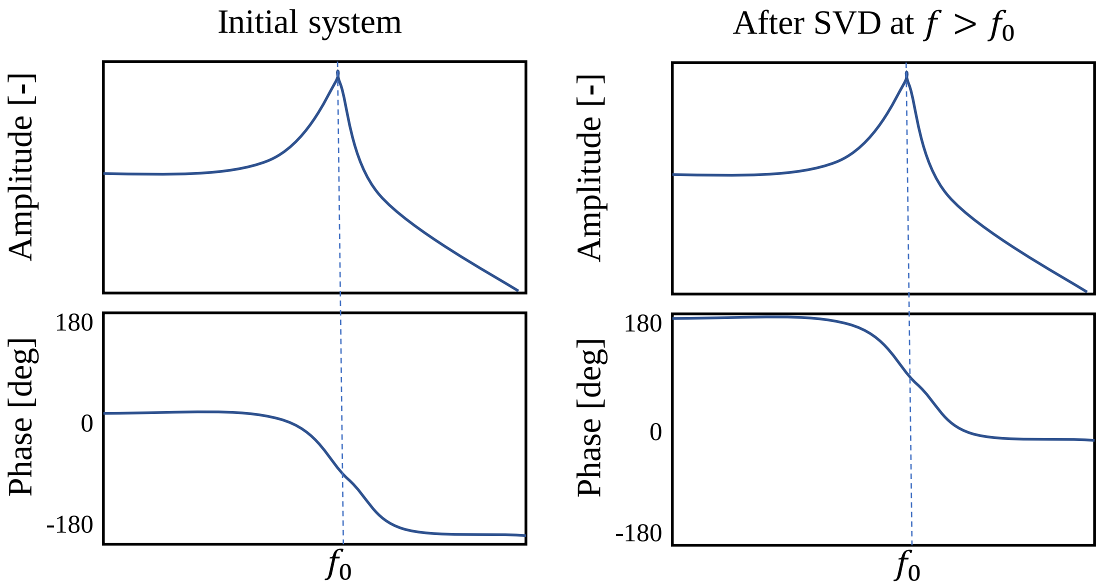
$$G_{decoupled} = U^{-1}(f_0)GV^{-H}(f_0)$$

And if you want to apply a decoupled controller to G (not $G_{decoupled}$), the controller will be

$$H_{dec} = V^{-H}(f_0)HU^{-1}(f_0) \tag{3}$$

Remark 1 if you evaluate the U and V matrix at a frequency where the system is already decoupled, the U and V matrix will be identity matrix and will not allow you to decouple your system. In most mechanical systems, this is the case at 0 Hz so it is not a good idea to opt for the steady-state decoupling.

Remark 2 from the model, it seems that it is safer to choose a frequency f_0 which is lower than the resonance frequencies of the system. Otherwise, the phase of the decoupled system might be shifted by 180° and the gain to use will have to be negative. This is not a major issue but when designing the controller, this has to be carefully taken into account. The phase shift arises from the fact that the singular value is a real positive value, i.e. the phase is 0. When you calculate the SVD at a frequency higher than a resonance, the phase of the signal below the resonance frequency will be shifted by 180°. Let's illustrate this, with the initial system show below. It's phase is 0 before the resonance frequency and -180° after. If the SVD matrix are evaluated above the resonance frequency, the phase of the resulting system after SVD will be 0 above the resonance frequency. Consequently, it will be 180° before the resonance frequency.



1.3 Eigenvalue decomposition

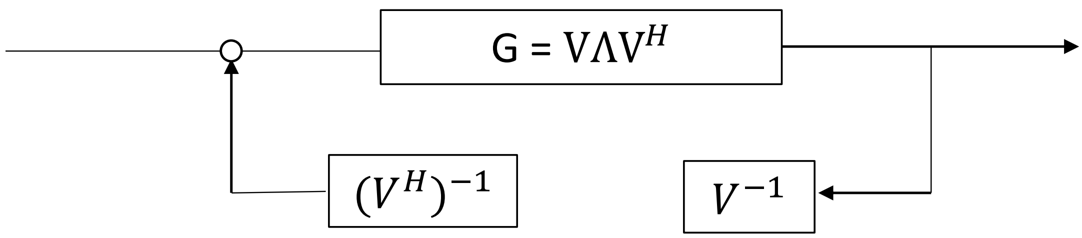
/!\This does not corresponds to the mode shapes of a mechanical system

To obtain the mode shapes, we should first project the system in the centralized coordinates and then calculate K/M because $M\ddot{x} = -Kx \to s^2X = -\frac{K}{M}X \to \omega^2 = K/M$

The mode shapes correspond to the resonance frequencies of the system while the eigenvalues are defined by

$$\begin{aligned} GV &= V\Lambda &(4)\\ GVV^H &= V\Lambda V^H &(5)\\ G &= V\Lambda V^H &(6) \end{aligned}$$

Note that eigenvalues can only be computed for square systems (p.75).



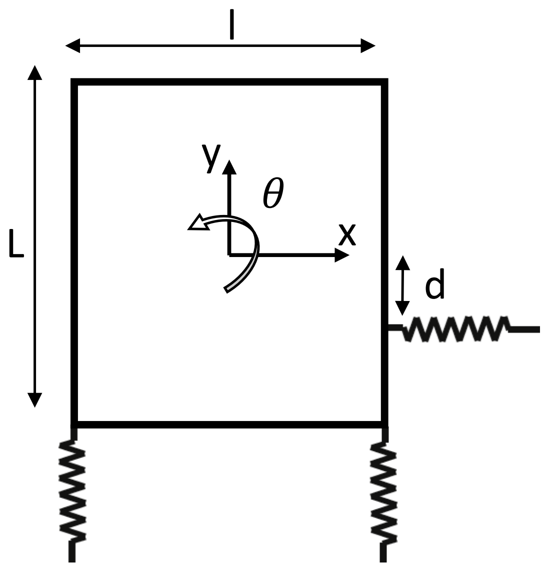
The eigenvalue decomposition can also be performed in three different ways, similarly to what is explained in the SVD section. The decoupled system is

$$G_{eigenvalue} = V^{-1}(f_0)GV^{-H}(f_0) \tag{7}$$

And if you want to apply a decoupled controller to G (not $G_{eigenvalue}$), the controller will be

$$H_{eig} = V^{-H}(f_0)HV^{-1}(f_0) \tag{8}$$

2 Application of the decomposition to a 3 d.o.f. model



The 3 dof model used to illustrate the different concepts discussed during the MIMO meetings is made of a rigid body which is supported by two pairs of vertical actuator and spring and one pair of horizontal actuator and spring. A sensor is collocated with each actuator.
The following sections will show if the decoupling using SVD will allow to improve the performance of inertial control in comparison to a decentralized approach.
Results of 4 group meetings: 26/10, 21/11, 28/11, 12/12

2.1 Singular values and eigenvalues decomposition

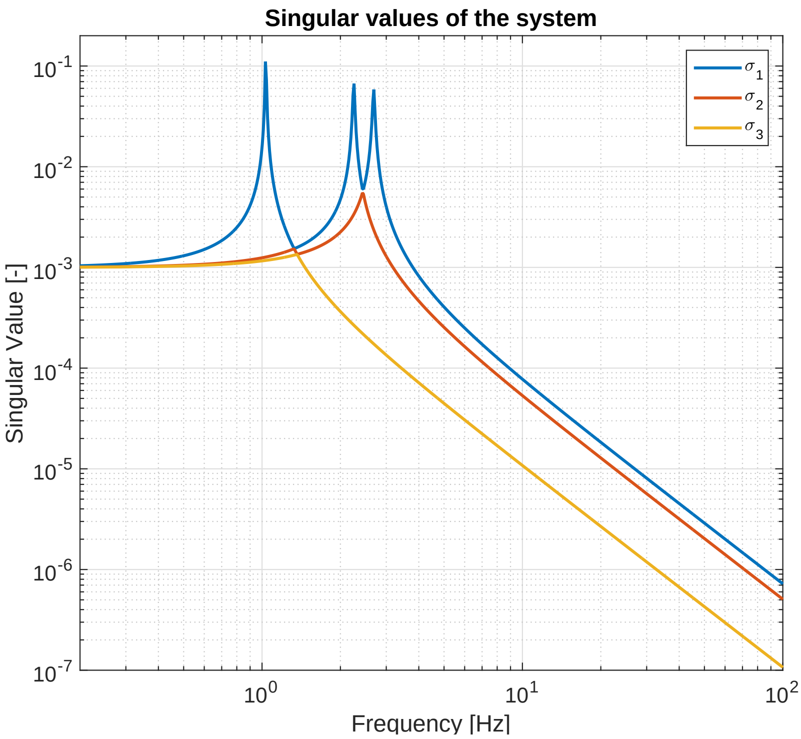
The two graphs below have been obtained thanks to a dynamic SVD and eigenvalue decoupling of the system in the decentralized coordinates. We can see that the frequencies of the peaks correspond exactly to the resonance frequencies of the system in the three d.o.f.:

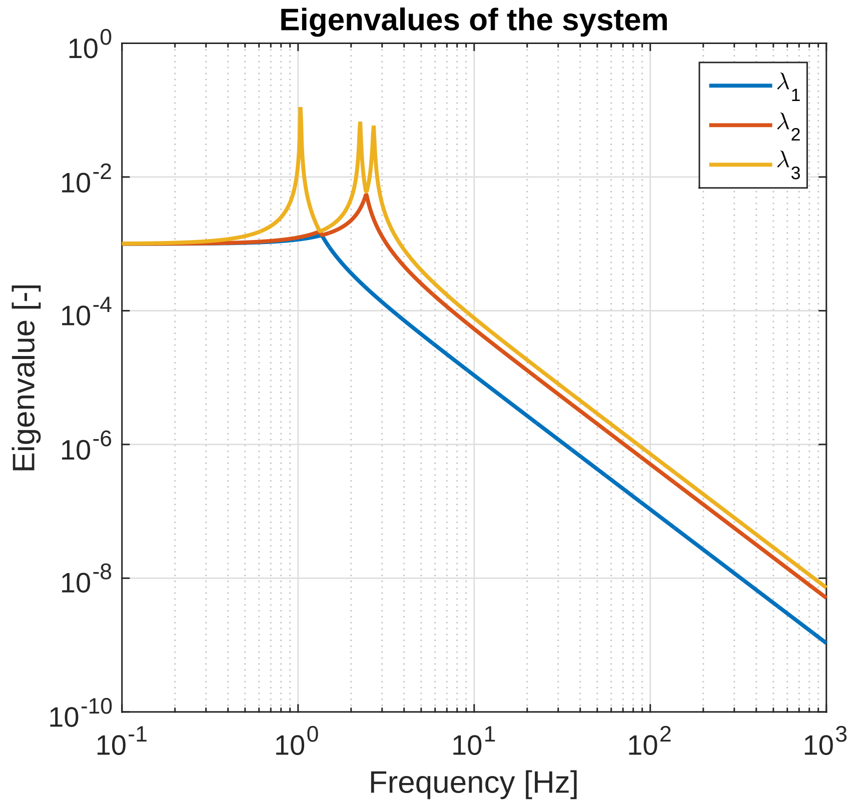
	1^{st} mode	2^{nd} mode	3^{rd} mode
Frequency	1.03 Hz	2.25 Hz	2.68 Hz

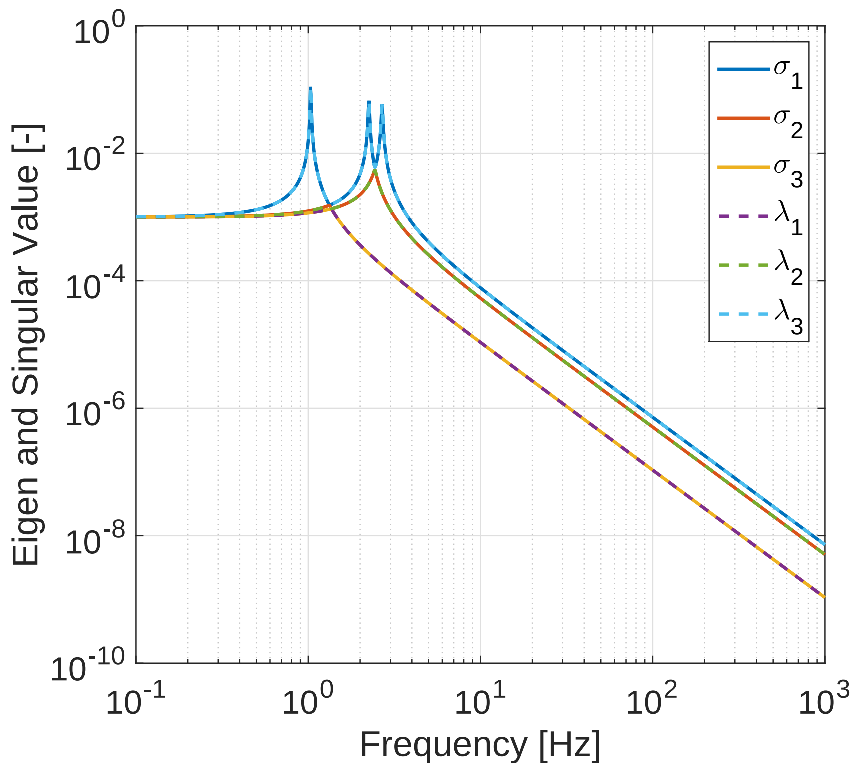
Table 1: Frequency corresponding the the different modes of the system.

Consequently, in both cases, the new system looks like three SISO models (except that σ_1 is always the largest singular value because the function sorts them). The last figure of this section shows the superposition of the singular values and eigenvalues. In this system which is a square system, the singular values and eigenvalues are identical.

In the rest of this report, we are only going to consider singular values.

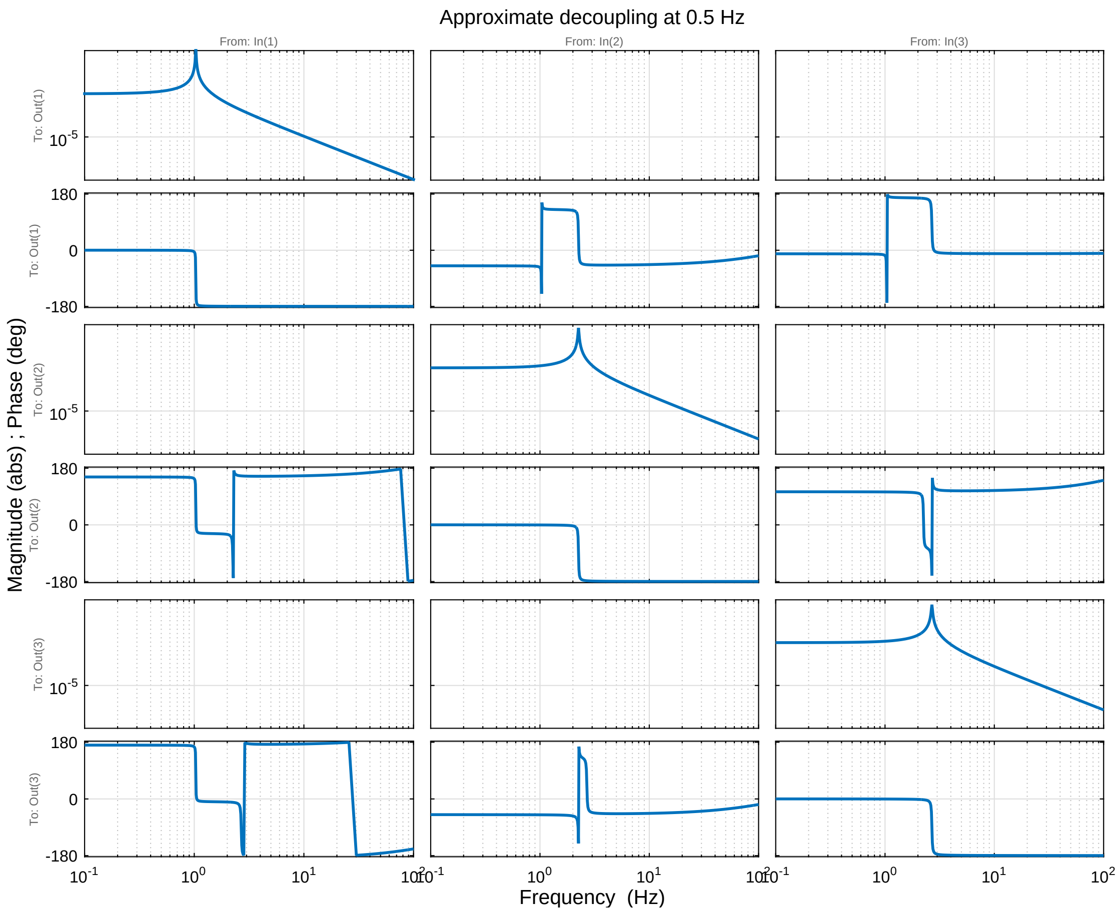



Eigenvalues of the system
Eigenvalue [-]
Frequency [Hz]
10^0
10^-2
10^-4
10^-6
10^-8
10^-10
10^-1
10^0
10^1
10^2
10^3
λ1
λ2
λ3


Eigen and Singular Value [-]
Frequency [Hz]
10^0
10^-2
10^-4
10^-6
10^-8
10^-10
10^-1
10^0
10^1
10^2
10^3
σ1
σ2
σ3
λ1
λ2
λ3

2.2 Decoupled system

The projection of the decentralized system in the SVD frame using the approximate SVD at 0.5 Hz is shown here. The system is decoupled as the maximum amplitude of the non-diagonal elements is on the order of 10^{-18}. Note that depending on the MATLAB function used, some strange behaviour appears. After some tests, we concluded that we should always prefer to use the ss and zpk functions and leave the tf function. Now let's see if this decoupling will help to improve the performance of the inertial control.



2.3 Control performance using a SISO approach

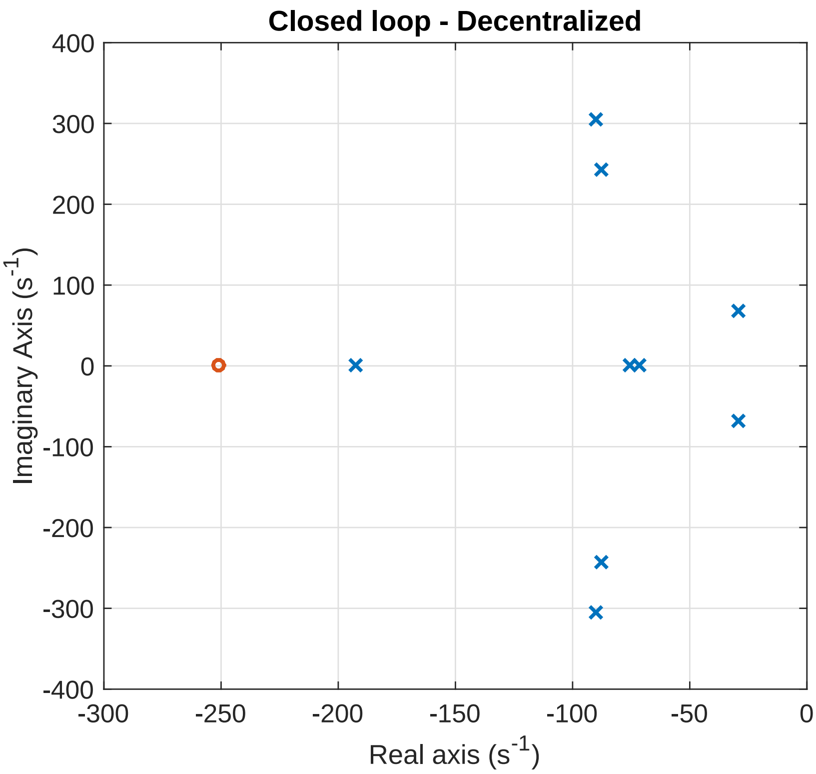
To design a robust controller for inertial control, the following steps should be carefully assessed:

1. Multiply the plant by the gain and show the open loop.
2. Verify the stability margins on the plot. If needed design the appropriate lead and lag to improve the margins. These can be different for each degree of freedom of the system.
3. Verify the poles and zeros locations of the closed loop: in order to verify the robustness of the control law over a range of gains, this pole-zero evaluation has to be performed with different values of the gain. The controller designed will have to be adapted depending on the gain as the unitary gain changes. This gain variation is necessary as in practice, the gain applied will never be exactly the same or it might fluctuate with time.

The goal of this section is to compare the performance of an inertial control based on a decentralized action to the one based on a SVD decoupling action. The SVD decoupling control was applied based on the definition made in the first section. In order to represent a realistic case, the gain of the inertial control has been chosen to reduce of a factor 100 the motion of the system in all direction. The corresponding gain is 10^5. The resulting open loop allow to verify if a lead and/or a lag is needed. In principle, the controller can be different for each diagonal element. Here, as the unitary gain of all diagonal element is around 20 Hz, the same lead has been applied

$$Lead = \frac{s + 2\pi 10}{s + 2\pi 40} \tag{9}$$

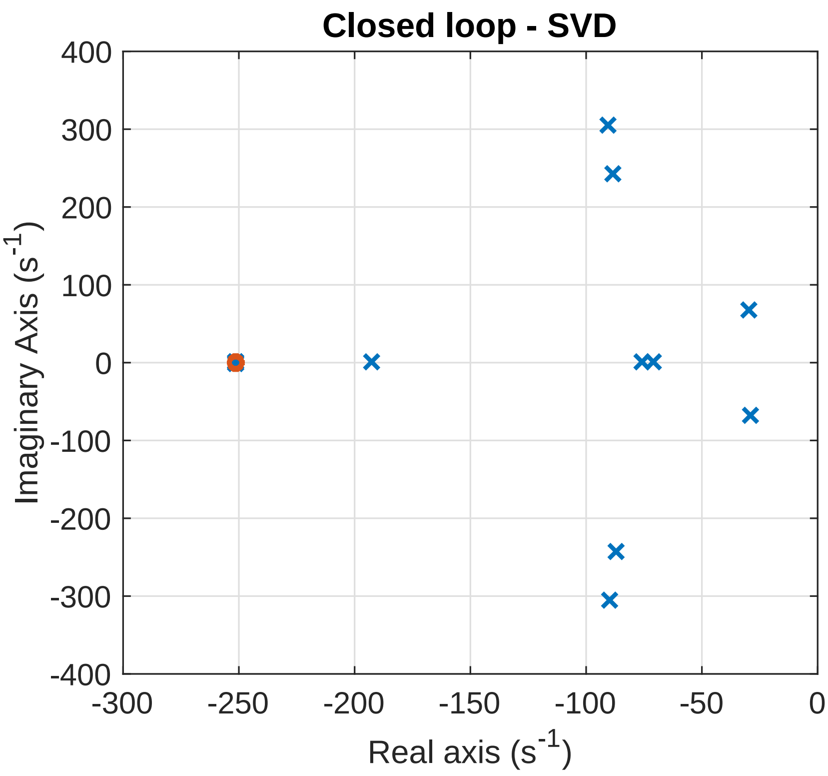
The same lead was used for both types of control as the unitary gain was crossed at the same frequency. The pole-zero map of the closed loop system for both cases is shown below. There is no positive real part pole or zero. Therefore, the robustness of the controller is guaranteed.

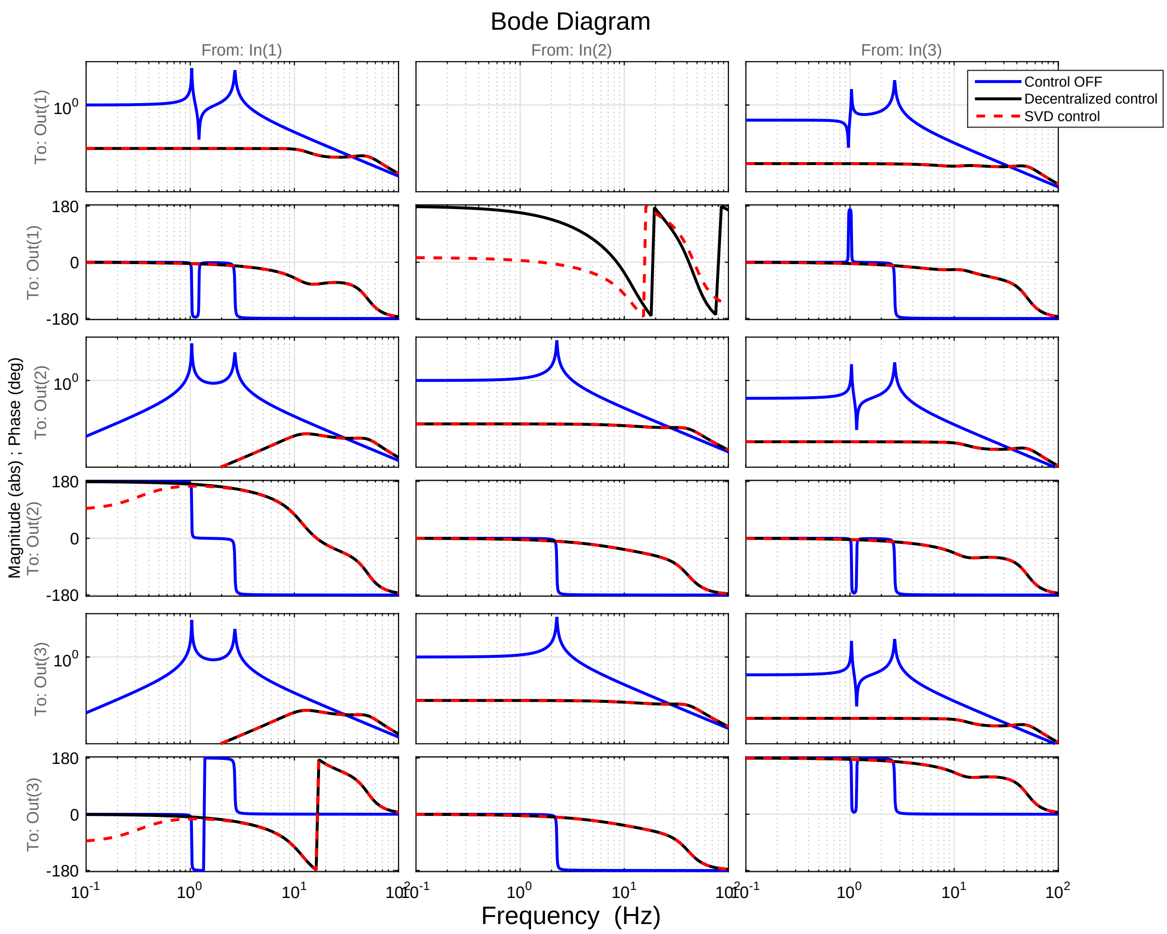


The performance of both controller is shown below. We can see that there is no clear difference between the two types of control laws in this application. It appears that in this case, the controllers are identical as the U and V matrix are unitary, we have

$$\begin{aligned}
H_{SVD} &= gV^{-H}(f_0)HU^{-1}(f_0) && (10)\\
&= gV^{-H}(f_0)\begin{pmatrix} Lead & 0 & 0 \\ 0 & Lead & 0 \\ 0 & 0 & Lead \end{pmatrix} U^{-1}(f_0) && (11)\\
&= gLeadV^{-H}(f_0)U^{-1}(f_0) && (12)\\
&= gLead = H_{decentralized} && (13)
\end{aligned}$$

The controllers will be different if the lead and/or the gain of the controller depend on the direction. For example, when the motion in one direction has to be more isolated, the decentralized control will not have the same performance as the above demonstration will not be applicable.


Closed loop - SVD
Imaginary Axis (s^-1)
Real axis (s^-1)
400
300
200
100
0
-100
-200
-300
-400
-300
-250
-200
-150
-100
-50
0


Bode Diagram
From: In(1)
From: In(2)
From: In(3)
To: Out(1)
To: Out(2)
To: Out(3)
Magnitude (abs) ; Phase (deg)
Control OFF
Decentralized control
SVD control
Frequency (Hz)

2.4 Singular value as a performance indicator

The bandwidth of the system corresponds to the frequency at which $\underline{\sigma}(GH)$ crosses 1. In fact, before the bandwidth frequency, GH is bigger than 1 and we isolate the system while above that frequency, GH is smaller than 1 and we don't isolate anymore.

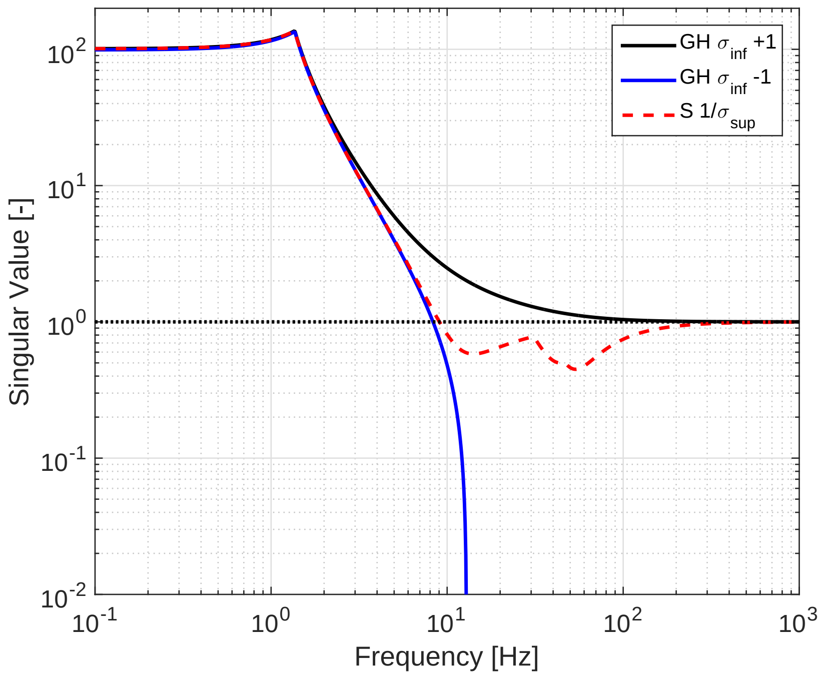
The singular values allow to have a first glimpse of the system performance. In fact, it has been stated (p.81) that

$$\underline{\sigma}(GH) - 1 \leq \frac{1}{\overline{\sigma}(S)} \leq \underline{\sigma}(GH) + 1 \tag{14}$$

Where S is the sensitivity function defined as $(1+\mathrm{GH})^{-1}$.

By evaluating the singular valuers of the open loop for all frequencies, we can predict the performance and the bandwidth of the closed loop system.

On the graph below, we can see that this assumption is verified in this example. Up to the resonance frequency, the inverse of the sensitivity is 100 which means that the system is isolated of a factor 100 in this frequency range. At the resonance frequency, the isolation is even bigger. Above the resonance frequency, the curves drops down. Therefore, the performance of the system will decrease. Finally around 10 Hz, the inverse of the sensitivity crosses 1 and by comparison with the performance curve above, we can see that the bandwidth is around 30 Hz.



Do we also have some information from the largest singular values of the open loop, i.e. is the following assumption also valid?

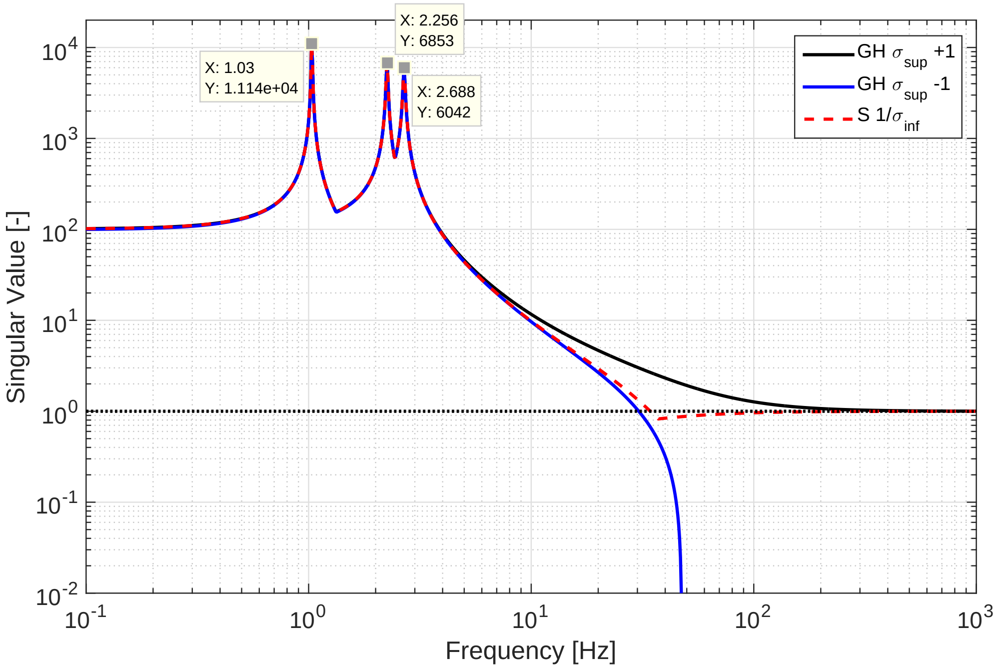
$$\overline{\sigma}(GH) - 1 \leq \frac{1}{\underline{\sigma}(S)} \leq \overline{\sigma}(GH) + 1 \tag{15}$$

If valid, can this assumption be used as a performance indicator?

The corresponding assumption is plotted on the graph below. The frequency of the three peaks corresponds to the three resonance frequencies of the system, see table 1. In addition, in this case, the amplitude below the resonance frequency is identical to the other case.

2.5 Meaning of the zeros of a MIMO system

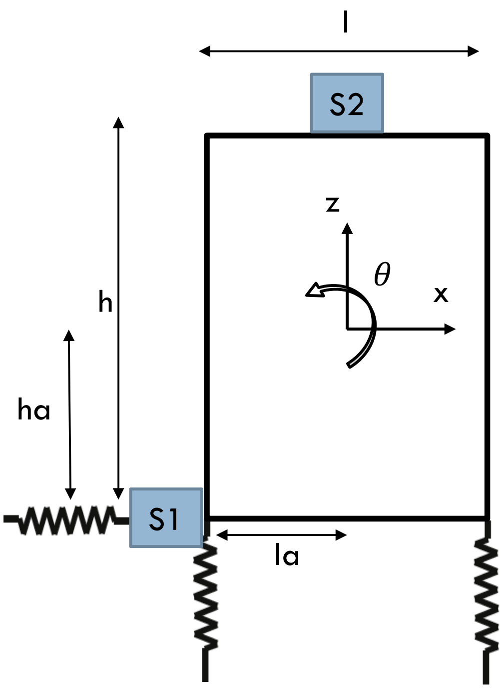
In a MIMO system, a zero corresponds to a zero in transmission [2]. It means that when some signal is injected, nothing is measured by the system. In other words, you are accumulating



energy in the system.

Methods to evaluate the transmission zeros of square and non-square systems are given in [2]. In addition, a MIMO system can be seen as a matrix of SISO transfer functions. In [2], it also shows that the zeros of these transfer functions does not systematically corresponds to the transmission zeros of the MIMO system.

3 Application to a 3 d.o.f. system with not the same number of input and outputs



To further investigate the role of SVD decomposition for the control performance of MIMO systems, the 3 dof system has been modified. The springs and actuators are located as before. One sensor is collocated with the horizontal spring/actuator pair. It measures the motion in the horizontal (x) and vertical (z) direction. A second sensor is placed on top of the rectangle, also measuring the motion in the horizontal and vertical direction. This second sensor is aligned with the centor of mass of the structure.
The system has three inputs and four outputs.

3.1 Singular value decomposition

Condition number [3]

3.2 Controller design and robustness study when using a MIMO approach

1. Do the SVD decomposition at the crossover frequency
2. Design a controller for each singular value
3. Check the performance of the resulting controller by evaluating the lowest singular value $\underline{\sigma}$ of the open loop.

4 Useful Matlab functions

ss: calculate the state-space model based on the A, B, C and D matrix specified by the user
zpk: returns a transfer function when the user specifies the location of the poles and zeros and the gain of the transfer function. This is the function to use to design a controller
tf: returns a transfer function when the user specifies the numerator and denominator polynoms. You better should use zpk of ss than tf to avoid numerical errors
minreal: evaluate the minimal realization of a system by pole-zero cancellation
feedback: calculate the closed-loop of the system, given the system, the controller and the input and output used to close the loop
evalfr and **freqresp**: evaluate a system at the frequency specified by the user
svd: calculate the Σ, U and V matrix of the system evaluated at a frequency (need to use evalfr or freqresp before)
eig: returns the eigenvalue and eigenvector of the system evaluated at a frequency (need to use evalfr or freqresp before)
sigma: evaluate the singular values and plot them in a figure
pzmap: returns the poles and zeros of a mimo system
pinv: allow to calculate the inverse of a matrix
B\ A : allow to calculate the division of the matrix A by the matrix A (for matrix, never do A/B)
robuststab: Calculate robust stability margins of uncertain multivariable system
ctrlpref: allow you to specify which are the default units for bode, Nyquist etc. It also allows to specify the label fontsize.

References

[1] Sigurd Skogestad and Ian Postlethwaite. *Multivariable feedback control: analysis and design*, volume 2. Wiley New York, 2007.

[2] William S. Levine. *The control handbook*. CRC press, 1996.

[3] Charlie Moore. Application of singular value decomposition to the design, analysis, and control of industrial processes. In *1986 American Control Conference*, pages 643–650. IEEE, 1986.